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CONTINUOUS ANALYSIS OF STRESSES FROM ARBITRARY SURFACE LOADS ON A HALF SPACE

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Abstract—A new form of elemental surface load on a half space is introduced, presuming a quasi-pyramidal variation of load which is doubly linear in each of four rectangular parts of a surface rectangle. Approximations of arbitrary load distributions by sums of such elements are continuous, piecewise linear in two directions and well adaptable. The loads may be normal or tangential. The explicit solutions obtained for all stress and displacement components due to each elemental load involve only elementary functions, are free of the discontinuities which arise with stepwise elements, and are suitable for computing. Some illustrative stress distributions are presented for elemental loads and for multiple pyramidal loads involving both normal and tangential loads. The value of the load continuity in the more complicated analyses of surface cracks is also illustrated.

INTRODUCTION

Many stress analyses involve effects due to forces on the surface of a half space distributed in patterns which may be simple or complex and which may be initially known or unknown. Analyses involving force patterns initially unknown include those for contact of elastic bodies, those for crack stresses to be freed on body surfaces and various ones for bodies loaded on more than one face. For some situations, such as Hertzian contact, the surface force distributions to be considered may be confined to some restricted general form; but for some situations the form must be very adaptable, such as for freeing loads in crack stress analysis. For the crack stress analysis there is also an advantage in keeping the freeing loads continuous since that allows an important checking process[1] and the property of continuity may well prove helpful as well as reasonable for many types of analysis. Thus the aim of the present theory is to treat effects from surface loads, both normal and tangential, in generality sufficient to consider arbitrary load distributions at least approximately without introducing artificial load discontinuities and their consequent distortion of stresses.

One method that has been used to treat an arbitrarily distributed normal load on a half space is to approximate its effects as being resultants of numerous point loads. This is the method applied, e.g. by Conry and Seirig in their study of elastic contact[2]. For some purposes use of point loads can suffice, despite its implied gross distortion of stresses on and near the surface, including even infinitudes in the applied load; but if the behavior of stresses at the surface is at issue then this approach is inadequate.

An early theory which allows treatment of arbitrarily distributed normal loads on a half space in terms of finite loads is that of Love[3], who considered effects of a uniform normal load acting on a surface rectangle. By adding effects from such loads on many rectangles with arbitrary individual intensities, one may treat effects from arbitrarily distributed pressure. However, at the borders of rectangles all three normal components of stress then may jump and at the corners of the rectangles one shear stress component may approach infinity logarithmically, as will be illustrated later. A parallel theory derived by Smith and Alavi[4] treats stresses due to uniform shear load on a rectangle. By adding the effects of such loads acting on many rectangles the effects of arbitrarily distributed shear loads can be treated, but where the surface load intensities jump a logarithmic singularity is implied either in two normal stress components or in one shear component, as is partially illustrated later. There may be analyses for which such discontinuities may be tolerable, but the discontinuities are almost always unrealistic and in some contexts they may be intolerable.

In order to get a continuous analysis of stresses due to an arbitrarily distributed pressure on a half space, Batra and Bell[5] considered effects due to normal loads linearly interpolable in both directions over a surface rectangle, with arbitrary intensity at each corner. By joining effects from many such loads meeting continuously at the edges of adjoining surface rectangles, it was found possible to approximate an arbitrary pressure distribution and yet imply only continuous stresses on the surface since singularities arising from loads on adjacent rectangles cancelled each other. In calculations using this theory the infinite stresses implied by some formulas along the edges of rectangles were to be eliminated by an appropriate bookkeeping procedure on the basis of their expected cancellation by other infinite stresses.

The present theory again seeks representation of arbitrary loads in continuous fashion by presuming linear interpolability in two directions, but two kinds of advance are made beyond the theory of Batra and Bell. In the first respect, a new kind of elemental load is employed which of itself eliminates all the discontinuities of stress components on the surface of the half space, and in doing so leads to stress formulas simpler than those of the earlier theory. In the

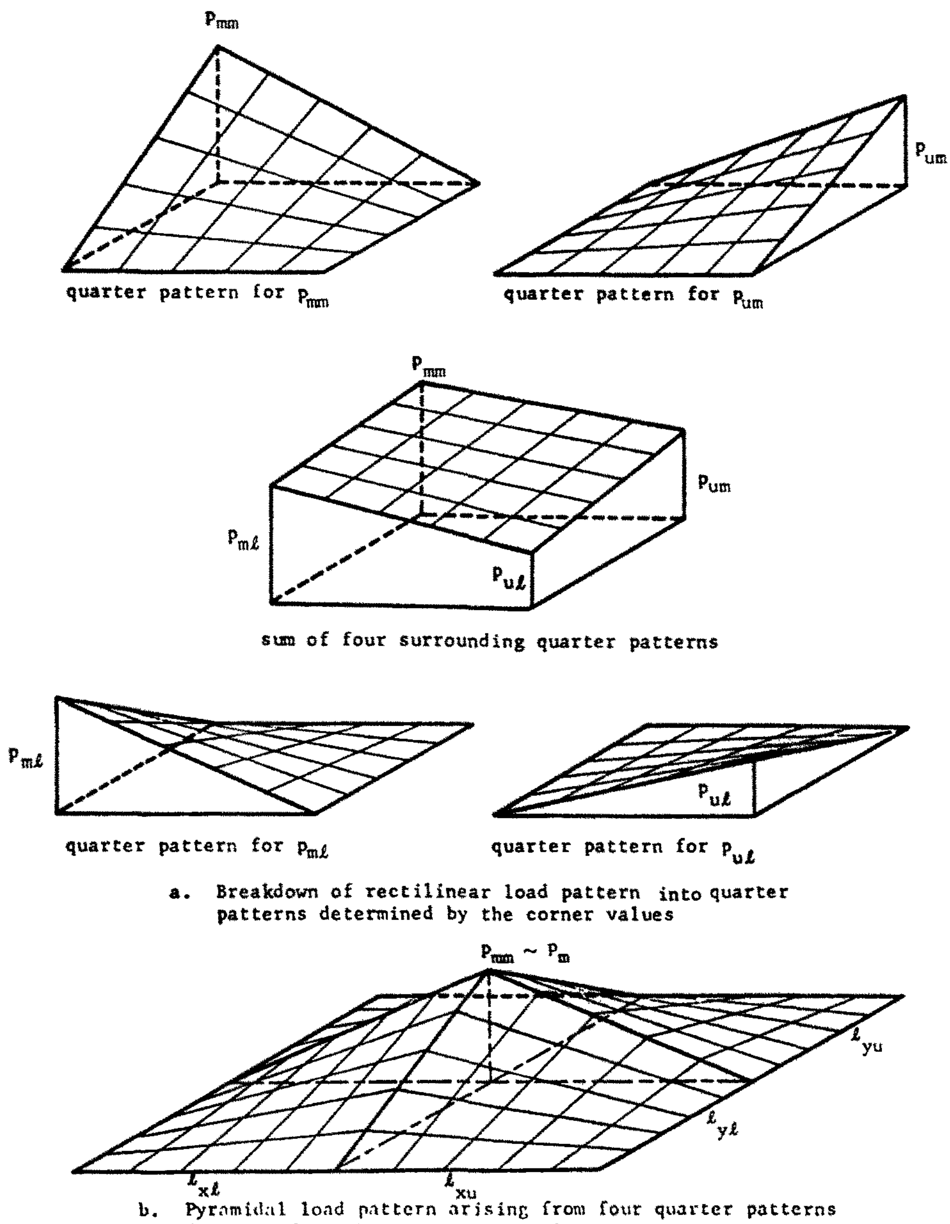


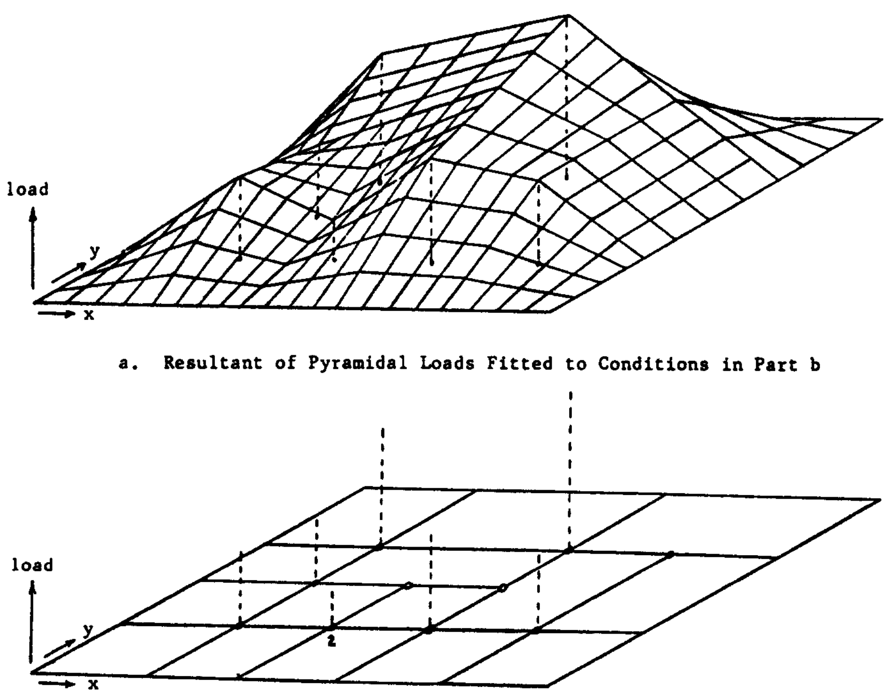
Fig. 1. Pyramidal load pattern providing load representation equivalent to that with doubly linear patterns on squares.

second respect, tangential as well as normal loads on the surface are considered, so that a unified treatment is provided embracing all the possible kinds of surface loads. The formulas for stresses and displacements to be derived here follow from theories already advanced by Lundberg[6] for generalized surface loads, but considerable organization is required in order to get the final concise results to be presented here.

CONCEPT OF PYRAMIDAL LOAD ELEMENTS

To understand the basis for choosing the form of load elements to be used here, whether they are for normal or tangential loads, one may begin with an element in which the load varies linearly in both directions over a rectangle, as shown by the central drawing in Fig. 1(a). Such a load can be decomposed into four parts such that in each part the load varies linearly in both directions and the only nonzero corner value is the same as a corner value in the original element, as is shown also in Fig. 1(a). If such a decomposition is applied to doubly linear load elements of four rectangles sharing a corner and having matching sides and if their components associated with the shared corner are combined, the resulting load distribution has the form shown in Fig. 1(b). Distributions of this form will be called pyramidal, since this roughly describes their shape, though the four partial distributions are really hyperboloidal. It should be noted that the four base rectangles may have quite different dimensions; all that is required in that regard is that the four base rectangles together form another rectangle.

If two or more pyramidal loads have overlapping bases with the same orientations, then over any shared rectangle their resultant load also varies linearly in both principal directions. Thus, with only mild restraint, one may subdivide an area over which an arbitrary load is to act into rectanglar parts of various dimensions and then assign arbitrary pyramidal elements which should be conformable to that load, subject only to the suitability of doubly linear interpolation within each rectangle. To get a full representation of possible load distributions, the subdivision into rectangles should normally allow placement of a pyramidal peak wherever two lattice lines fully cross each other. A scanning of a proposed subdivision can show whether it provides a base for a pyramidal load at each such crossing point. Of course, the overall load represented will vanish at the outer border of the set of rectangles, but if a steeply dropping load is desired then narrow rectangles can be used.



a. Resultant of Pyramidal Loads Fitted to Conditions in Part b

b. Representative Surface Lattice and Total Loads at Pyramid Points

Fig. 2. Illustration of a load represented by pyramidal elements.

Figure 2 provides an example of how an array of pyramidal load elements can represent a load distribution. Figure 2(b) shows a portion of a plane divided into rectangles and it can be seen that a pyramidal load element may have its peak at any point where two lattice lines fully cross, these points being marked by filled dots. At interior points where the lines meet but do not cross, as marked by open dots, pyramidal peaks are not required. Any load intensities might be assigned at the pyramid points (filled dots), but by taking those loads as shown in Fig. 2(b) one defines the interpolated load distribution shown in Fig. 2(a). If the smallest possible pyramidal bases are used, then at six of the pyramid points only one load element differs from zero, but at one point (marked "2") two load elements contribute. Inside all the rectangles, except those at the outside corners, two or more pyramidal load elements contribute. Combining seven such elements allows full fitting for any load distribution which is doubly interpolable in each rectangle of Fig. 2(b) and which vanishes along the outer edge.

DEVELOPMENT OF POTENTIAL FUNCTIONS

Using a rectangular coordinate system (x, y, z), with z being depth below the surface of a half space, consider a pyramidal load acting on a base defined by lines where $x = x_l$, x_m, or x_u and $y = y_l$, y_m, or y_u as shown in Fig. 3. In order to reserve x, y, and z as coordinates where stresses or displacements are to be evaluated, let ξ and η denote the x and y coordinates where a part of the load acts. Three forms of load are to be considered, namely, $p(\xi,\eta) \sim -\sigma_z(\xi,\eta,0)$, $s(\xi,\eta) \sim \tau_{zx}(\xi,\eta,0)$ and $t(\xi,\eta) \sim \tau_{yz}(\xi,\eta,0)$, but momentarily consider only a load $p(\xi,\eta)$ and let its maximum value be p_m where $\xi = x_m$ and $\eta = y_m$. It can be seen that over the four parts of the base it must be

$$p(\xi,\eta) = \begin{cases} p_m(x_u-\xi)(y_u-y)/(l_{xu}l_{yu}) \text{ in Area I,} \\ p_m(\xi-x_l)(y_u-\eta)/(l_{xl}l_{yu}) \text{ in Area II,} \\ p_m(\xi-x_l)(\eta-y_l)/(l_{xl}l_{yl}) \text{ in Area III,} \\ p_m(x_u-\xi)(\eta-y_l)/(l_{xu}l_{yl}) \text{ in Area IV,} \end{cases} \tag{1}$$

and it must vanish beyond these areas. Here l_{xl}, l_{xu}, l_{yl}, and l_{yu} are dimensions of the component rectangles as shown in Fig. 3. It is advantageous to employ x and y components of distance from a point (x, y, z) where effects are felt to the point $(\xi, \eta, 0)$ where load is acting, so let them be

$$\alpha = \xi - x, \ \beta = \eta - y, \tag{2}$$

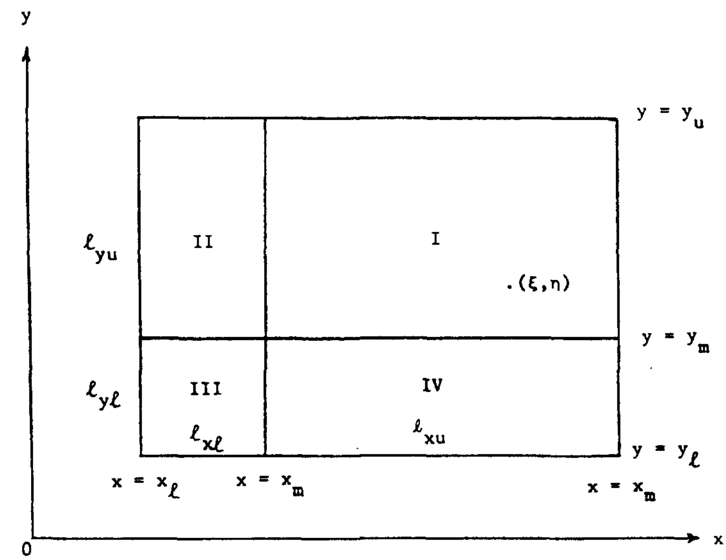


Fig. 3. Notation used in defining bases for pyramidal loads.

and let α_l, α_u, β_l, and β_u be their values when ξ and η are at the extremities of the base of the pyramidal load, that is

$$\alpha_l = x_l - x,\ \alpha_u = x_u - x,\ \beta_l = y_l - y,\ \beta_u = y_u - y. \tag{3}$$

Then again the load distributions over the four parts of the base are

$$p(\xi, \eta) \equiv \bar{p}(\alpha, \beta) = \begin{cases} p_m(\alpha_u - \alpha)(\beta_u - \beta)/(l_{xu} l_{yu}) & \text{in Area I,} \\ p_m(\alpha - \alpha_l)(\beta_u - \beta)/(l_{xl} l_{yu}) & \text{in Area II,} \\ p_m(\alpha - \alpha_l)(\beta - \beta_l)/(l_{xl} l_{yl}) & \text{in Area III,} \\ p_m(\alpha_u - \alpha)(\beta - \beta_l)/(l_{xu} l_{yl}) & \text{in Area IV.} \end{cases} \tag{4}$$

Similar expressions hold for tangential loads $s(\xi, \eta) \equiv \bar{s}(\alpha, \beta)$ and $t(\xi, \eta) \equiv \bar{t}(\alpha, \beta)$, but with peak values s_m or t_m replacing p_m.

In his analysis of stresses due to loads arbitrarily distributed on a half space, Lundberg[6] introduced the following functions:

$$\begin{aligned} L(x, y, z, \xi, \eta) &= -\frac{1}{2\pi}(\alpha^2 + \beta^2 + z^2)^{-1/2}, \\ M(x, y, z, \xi, \eta) &= \int_z^{+\infty} \frac{\partial L}{\partial x}\,\mathrm{d}z;\ N(x, y, z, \xi, \eta) = \int_z^{\infty} \frac{\partial L}{\partial y}\,\mathrm{d}z, \end{aligned} \tag{5}$$

and he showed they could be used advantageously in expressing these potential functions:

$$\begin{aligned} V(x, y, z) &= -\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [L \cdot p(\xi, \eta) + M \cdot s(\xi, \eta) + N \cdot t(\xi, \eta)]\,\mathrm{d}\xi\,\mathrm{d}\eta \\ S(x, y, z) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} L \cdot s(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,\ T(x, y, z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} L \cdot t(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{6}$$

He further showed how all the stress and displacement components can be expressed in terms of assorted (and often multiple) derivatives and integrals of V, S and T, so in principle all that needs to be done is to perform the steps he indicated using loads of the form shown in the eqn (4), but the algebra involved is burdensome and occasionally devious.

Evaluation of M and N shows that V, S and T here have the forms

$$\begin{aligned} V &= \frac{1}{4\pi}\int_{\beta_l}^{\beta_u}\int_{\alpha_l}^{\alpha_u}\left\{\frac{\bar{p}(\alpha, \beta)}{\sqrt{(\alpha^2 + \beta^2 + z^2)}} + \frac{\alpha\bar{s}(\alpha, \beta) + \beta\bar{t}(\alpha, \beta)}{\alpha^2 + \beta^2}\left[1 - \frac{z}{\sqrt{(\alpha^2 + \beta^2 + z^2)}}\right]\right\}\mathrm{d}\alpha\,\mathrm{d}\beta \\ S &= \frac{-1}{2\pi}\int_{\beta_l}^{\beta_u}\int_{\alpha_l}^{\alpha_u}\frac{\bar{s}(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\sqrt{(\alpha^2 + \beta^2 + z^2)}},\ T = \frac{-1}{2\pi}\int_{\beta_l}^{\beta_u}\int_{\alpha_l}^{\alpha_u}\frac{\bar{t}(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\sqrt{(\alpha^2 + \beta^2 + z^2)}}. \end{aligned} \tag{7}$$

Since $\bar{p}(\alpha, \beta)$, $\bar{s}(\alpha, \beta)$ and $\bar{t}(\alpha, \beta)$ all involve only terms which are constants multiplied by 1, α, β or $\alpha\beta$, there are only twelve integrations needed for the eqns (7). In the integration it is convenient to put

$$\begin{aligned} \rho &= \sqrt{(\alpha^2 + \beta^2 + z^2)} \\ \theta_1 &= \arctan\frac{\beta}{\alpha} - \arctan\frac{\beta z}{\alpha\rho}, \\ \theta_2 &= \arctan\frac{\alpha}{\beta} - \arctan\frac{\alpha z}{\beta\rho}, \\ \theta_3 &= \arctan\frac{\alpha\beta}{z\rho}, \end{aligned} \tag{8}$$

and to use principal values for the arctangents (that is $> -\pi/2$ and $\leq \pi/2$). Then, using some formulas given by Batra and Bell in their appendix†, one can show that in their indefinite forms the integrals needed for the eqn (7) are

$$
\begin{aligned}
g_{11}(\alpha,\beta,z) &\equiv \int\!\!\int \frac{\mathrm{d}\alpha\,\mathrm{d}\beta}{\rho} = \beta\ln(\alpha+\rho)+\alpha\ln(\beta+\rho)-z\theta_3,\\
g_{12}(\alpha,\beta,z) &\equiv \int\!\!\int \frac{\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\rho} = \frac{\beta\rho}{2}+\frac{\alpha^2+z^2}{2}\ln(\beta+\rho),\\
g_{13}(\alpha,\beta,z) &\equiv \int\!\!\int \frac{\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\rho} = \frac{\alpha\rho}{2}+\frac{\beta^2+z^2}{2}\ln(\alpha+\rho),\\
g_{14}(\alpha,\beta,z) &\equiv \int\!\!\int \frac{\alpha\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\rho} = \rho^3/3,\\
g_{21}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \beta\ln(z+\rho)+z\ln(\beta+\rho)+\alpha\theta_1,\\
g_{22}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\alpha^2\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \frac{\alpha\beta}{2}-\beta z\ln(\alpha+\rho)+\frac{\alpha^2}{2}\theta_1-\frac{\beta^2}{2}\theta_2+\frac{z^2}{2}\theta_3,\\
g_{23}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\alpha\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \frac{\alpha^2+\beta^2}{2}\ln(z+\rho)+\frac{z\rho}{2},\\
g_{24}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\alpha^2\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \frac{\alpha\beta^2}{3}+\frac{\alpha z\rho}{6}+\frac{\alpha^3}{3}\ln(z+\rho)-\frac{z}{6}(3\beta^2+z^2)\ln(\alpha+\rho)-\frac{\beta^3}{3}\theta_2,\\
g_{31}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \alpha\ln(z+\rho)+z\ln(\alpha+\rho)+\beta\theta_2,\\
g_{32}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\alpha\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \frac{\alpha^2+\beta^2}{2}\ln(z+\rho)+\frac{z\rho}{2},\\
g_{33}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\beta^2\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \frac{\alpha\beta}{2}-\alpha z\ln(\beta+\rho)-\frac{\alpha^2}{2}\theta_1+\frac{\beta^2}{2}\theta_2+\frac{z^2}{2}\theta_3,\\
g_{34}(\alpha,\beta,z) &\equiv \int\!\!\int \left(1-\frac{z}{\rho}\right)\frac{\alpha\beta^2\,\mathrm{d}\alpha\,\mathrm{d}\beta}{\alpha^2+\beta^2} = \frac{\alpha^2\beta}{3}+\frac{\beta z\rho}{6}+\frac{\beta^3}{3}\ln(z+\rho)-\frac{z}{6}(3\alpha^2+z^2)\ln(\beta+\rho)-\frac{\alpha^3}{3}\theta_1.
\end{aligned}
\tag{9}
$$

Here terms lacking either α or β have been dropped, since they would drop when limits are applied to the integrals as required by the equations (7).

Let $V^{(1)}$ be the parts of V contributed by the normal loads. Then letting also $\alpha_m = x_m - x$ and $\beta_m = y_m - y$, the contributions to $V^{(1)}$ from the loads on the four component rectangles are

$$
\begin{aligned}
V_I^{(1)} = \frac{p_m}{4\pi l_{xu} l_{yu}}\{&\alpha_u\beta_u[g_{11}(\alpha_u,\beta_u,z)-g_{11}(\alpha_u,\beta_m,z)-g_{11}(\alpha_m,\beta_u,z)+g_{11}(\alpha_m,\beta_m,z)]\\
&-\beta_u[g_{12}(\alpha_u,\beta_u,z)-g_{12}(\alpha_u,\beta_m,z)-g_{12}(\alpha_m,\beta_u,z)+g_{12}(\alpha_m,\beta_m,z)]\\
&-\alpha_u[g_{13}(\alpha_u,\beta_u,z)-g_{13}(\alpha_u,\beta_m,z)-g_{13}(\alpha_m,\beta_u,z)+g_{13}(\alpha_m,\beta_m,z)]\\
&+[g_{14}(\alpha_u,\beta_u,z)-g_{14}(\alpha_u,\beta_u,z)-g_{14}(\alpha_m,\beta_u,z)+g_{14}(\alpha_m,\beta_m,z)]\\
V_{II}^{(1)} = \frac{p_m}{4\pi l_{xl} l_{yu}}\{&-\alpha_l\beta_u[g_{11}(\alpha_m,\beta_u,z)-g_{11}(\alpha_m,\beta_m,z)-g_{11}(\alpha_l,\beta_u,z)+g_{11}(\alpha_l,\beta_m,z)]\\
&+\beta_u[g_{12}(\alpha_m,\beta_u,z)-g_{12}(\alpha_m,\beta_m,z)-g_{12}(\alpha_l,\beta_u,z)+g_{12}(\alpha_l,\beta_m,z)]\\
&+\alpha_l[g_{13}(\alpha_m,\beta_u,z)-g_{13}(\alpha_m,\beta_m,z)-g_{13}(\alpha_l,\beta_u,z)+g_{13}(\alpha_l,\beta_m,z)]\\
&-[g_{14}(\alpha_m,\beta_u,z)-g_{14}(\alpha_m,\beta_m,z)-g_{14}(\alpha_l,\beta_u,z)+g_{14}(\alpha_l,\beta_m,z)]\},
\end{aligned}
\tag{10}
$$

†The formulas provided there are summarized here in Appendix A.

$$V_{III}^{(1)} = \frac{p_m}{4\pi l_{xl} l_{yl}} \{\alpha_l \beta_l [g_{11}(\alpha_m, \beta_m, z) - g_{11}(\alpha_m, \beta_l, z) - g_{11}(\alpha_l, \beta_m, z) + g_{11}(\alpha_l, \beta_l, z)]$$
$$-\beta_l [g_{12}(\alpha_m, \beta_m, z) - g_{12}(\alpha_m, \beta_l, z) - g_{12}(\alpha_l, \beta_m, z) + g_{12}(\alpha_l, \beta_l, z)]$$
$$-\alpha_l [g_{13}(\alpha_m, \beta_m, z) - g_{13}(\alpha_m, \beta_l, z) - g_{13}(\alpha_l, \beta_m, z) + g_{13}(\alpha_l, \beta_l, z)]$$
$$+ [g_{14}(\alpha_m, \beta_m, z) - g_{14}(\alpha_m, \beta_l, z) - g_{14}(\alpha_l, \beta_m, z) + g_{14}(\alpha_l, \beta_l, z)],$$

$$V_{IV}^{(1)} = \frac{p_m}{4\pi l_{xu} l_{yl}} \{-\alpha_u \beta_l [g_{11}(\alpha_u, \beta_m, z) - g_{11}(\alpha_u, \beta_l, z) - g_{11}(\alpha_m, \beta_m, z) + g_{11}(\alpha_m, \beta_l, z)]$$
$$+ \beta_l [g_{12}(\alpha_u, \beta_m, z) - g_{12}(\alpha_u, \beta_l, z) - g_{12}(\alpha_m, \beta_m, z) + g_{12}(\alpha_m, \beta_l, z)]$$
$$+ \alpha_u [g_{13}(\alpha_u, \beta_m, z) - g_{13}(\alpha_u, \beta_l, z) - g_{13}(\alpha_m, \beta_m, z) + g_{13}(\alpha_m, \beta_l, z)]$$
$$- [g_{14}(\alpha_u, \beta_m, z) - g_{14}(\alpha_u, \beta_l, z) - g_{14}(\alpha_m, \beta_m, z) + g_{14}(\alpha_m, \beta_l, z)]\}.$$

The required summation of these parts is aided by facts such as

$$\frac{\alpha_l}{l_{xl}} + \frac{\alpha_u}{l_{xu}} = \frac{(x_l - x_m) + (x_m - x)}{x_m - x_l} + \frac{(x_u - x_m) + (x_m - x)}{x_u - x_m} = \alpha_m \left(\frac{1}{l_{xl}} + \frac{1}{l_{xm}}\right).$$

Accounting for such relations, one finds from the addition of all the parts of $V^{(1)}$ that

$$V^{(1)} \equiv V_I^{(1)} + V_{II}^{(1)} + V_{III}^{(1)} + V_{IV}^{(1)}$$
$$= \frac{p_m}{4\pi} [\alpha\beta g_{11}(\alpha, \beta, z) - \beta g_{12}(\alpha, \beta, z) - \alpha g_{13}(\alpha, \beta, z) + g_{14}(\alpha, \beta, z)]^*, \tag{11}$$

where the asterisk denotes the following weighted sum of evaluations of any function $F(\alpha, \beta, z)$ at the nine corners in the pyramidal base:

$$[F(\alpha, \beta, z)]^* = +\frac{1}{l_{xl} l_{yu}} F(\alpha_l, \beta_u, z) - \left(\frac{1}{l_{xl}} + \frac{1}{l_{xu}}\right) \frac{1}{l_{yu}} F(\alpha_m, \beta_u, z) + \frac{1}{l_{xu} l_{yu}} F(\alpha_u, \beta_u, z)$$
$$-\frac{1}{l_{xl}} \left(\frac{1}{l_{yl}} + \frac{1}{l_{yu}}\right) F(\alpha_l, \beta_m, z) + \left(\frac{1}{l_{xl}} + \frac{1}{l_{xu}}\right)\left(\frac{1}{l_{yl}} + \frac{1}{l_{yu}}\right) F(\alpha_m, \beta_m, z)$$
$$-\frac{1}{l_{xu}} \left(\frac{1}{l_{yl}} + \frac{1}{l_{yu}}\right) F(\alpha_u, \beta_m, z) \tag{12}$$
$$+\frac{1}{l_{xl} l_{yl}} F(\alpha_l, \beta_l, z) - \left(\frac{1}{l_{xl}} + \frac{1}{l_{xu}}\right) \frac{1}{l_{yl}} F(\alpha_m, \beta_l, z) + \frac{1}{l_{xu} l_{yl}} F(\alpha_u, \beta_l, z).$$

It can be shown that terms of any $F(\alpha, \beta, z)$ which are constant or linear in either α or β cancel in this sum. Thus it has been shown that if one further defines

$$g_1(\alpha, \beta, z) = \alpha\beta g_{11}(\alpha, \beta, z) - \beta g_{12}(\alpha, \beta, z) - \alpha g_{13}(\alpha, \beta, z) + g_{14}(\alpha, \beta, z)$$
$$= \frac{\beta^2 - z^2}{6} [3\alpha \ln(\alpha + \rho) - \rho] + \frac{\alpha^2 - z^2}{6} [3\beta \ln(\beta + \rho) - \rho] - \alpha\beta z \theta_3, \tag{13}$$

then

$$V^{(1)} = \frac{p_m}{4\pi} [g_1(\alpha, \beta, z)]^*. \tag{14}$$

Since the grouping of terms from the eqns (10) leading to eqn (11) used no property of the $g_{1j}(\alpha, \beta, z)$ other than their evaluability, it can be seen that it is useful also to introduce the functions

$$g_2(\alpha, \beta, z) = \alpha\beta g_{21}(\alpha, \beta, z) - \beta g_{22}(\alpha, \beta, z) - \alpha g_{23}(\alpha, \beta, z) + g_{24}(\alpha, \beta, z)$$
$$= \frac{z}{6}(3\beta^2 - z^2) \ln(\alpha + \rho) + \alpha\beta z \ln(\beta + \rho) - \frac{\alpha}{6}(\alpha^2 - 3\beta^2) \ln(z + \rho) \tag{15}$$
$$+ \frac{\alpha^2 \beta}{2} \theta_1 + \frac{\beta^3}{6} \theta_2 - \frac{\beta z^2}{2} \theta_3 - \frac{\alpha z \rho}{3} - \frac{\alpha \beta^2}{6},$$

and

$$g_3(\alpha,\beta,z)=\alpha\beta g_{31}(\alpha,\beta,z)-\beta g_{32}(\alpha,\beta,z)-\alpha g_{33}(\alpha,\beta,z)+g_{34}(\alpha,\beta,z)$$

$$=\alpha\beta z\ \ln(\alpha+\rho)+\frac{z}{6}(3\alpha^2-z^2)\ln(\beta+\rho)+\frac{\beta}{6}(3\alpha^2-\beta^2\ln(z+\rho) \quad (16)$$

$$+\frac{\alpha^3}{6}\theta_1+\frac{\alpha\beta^2}{2}\theta_2-\frac{\alpha z^2}{2}\theta_3-\frac{\alpha^2\beta}{6}-\frac{\beta z\rho}{3}.$$

Then also it follows that the potential functions are

$$V(x,y,z)=\frac{1}{4\pi}[p_m g_1(\alpha,\beta,z)+s_m g_2(\alpha,\beta,z)+t_m g_3(\alpha,\beta,z)]^*,$$

$$S(x,y,z)=-\frac{s_m}{2\pi}[g_1(\alpha,\beta,z)]^*,\ T(x,y,z)=-\frac{t_m}{2\pi}[g_1(\alpha,\beta,z)]^*. \quad (17)$$

Considering their origins, these functions are remarkably compact. Compactness is important for the voluminous steps yet to be taken. (One may note also that the terms $\alpha\beta^2/6$ in g_2 and $\alpha^2\beta/6$ in g_3 may be dropped from eqn (17) because of their linearity in α or β. Such deletions are often helpful in the steps remaining.)

STRESSES AND DISPLACEMENTS DUE TO PYRAMIDAL LOADS

To systematize the derivations when three kinds of pyramidal loads can be applied simultaneously over the same base, let

$$p_m=c_1,\ s_m=c_2,\ t_m=c_3. \quad (18)$$

Then it has been shown that

$$2V=\frac{1}{2\pi}\left[\sum_{i=1}^{3}c_i g_i(\alpha,\beta,z)\right]^*,$$

$$S=-\frac{1}{2\pi}[c_2 g_1(\alpha,\beta,z)]^*,\ T=-\frac{1}{2\pi}[c_3 g_1(\alpha,\beta,z)]^*. \quad (19)$$

It may be observed that

$$\frac{\partial}{\partial x}[g_i(\alpha,\beta,z)]^*=-\left[\frac{\partial g_i}{\partial\alpha}\right]^*,\ \frac{\partial}{\partial y}[g_i(\alpha,\beta,z)]^*=-\left[\frac{\partial g_i}{\partial\beta}\right],\ \frac{\partial}{\partial z}[g_i(\alpha,\beta,z)]^*=\left[\frac{\partial g_i}{\partial z}\right]^*. \quad (20)$$

Then, from Lundberg's eqn (6), the stresses and displacements throughout the half space are given by:

$$
\left.
\begin{aligned}
\sigma_x &= \frac{-1}{2\pi}\Bigg[\sum_i c_i\left\{z\frac{\partial^2 g_i}{\partial\alpha^2}-(1-2\nu)\int_z^\infty \frac{\partial^2 g_i}{\partial\alpha^2}\,\mathrm{d}z-2\nu\frac{\partial g_i}{\partial z}\right\}-c_2\frac{\partial g_1}{\partial\alpha}-\\
&\qquad \int_z^\infty\!\!\int_z^\infty \frac{\partial^2}{\partial\alpha\partial\beta}\left(c_3\frac{\partial g_1}{\partial\alpha}-c_2\frac{\partial g_1}{\partial\alpha}\right)\mathrm{d}^2z\Bigg]^*,\\
\sigma_y &= \frac{-1}{2\pi}\Bigg[\sum_i c_i\left\{z\frac{\partial^2 g_i}{\partial\beta^2}-(1-2\nu)\int_z^\infty \frac{\partial^2 g_i}{\partial\beta^2}\,\mathrm{d}z-2\nu\frac{\partial g_i}{\partial z}\right\}-c_3\frac{\partial g_i}{\partial\beta}-\\
&\qquad \int_z^\infty\!\!\int_z^\infty \frac{\partial^2}{\partial\alpha\partial\beta}\left(c_2\frac{\partial g_1}{\partial\beta}-c_3\frac{\partial g_1}{\partial\beta}\right)\mathrm{d}^2z\Bigg]^*,\\
\sigma_z &= \frac{-1}{2\pi}\left[\sum_i c_i\left\{z\frac{\partial^2 g_i}{\partial\beta^2}-\frac{\partial g_i}{\partial z}\right\}+c_2\frac{\partial g_i}{\partial\alpha}+c_3\frac{\partial g_1}{\partial\beta}\right]^*,\\
\tau_{xy} &= \frac{-1}{2\pi}\Bigg[\sum_i c_i\left\{z\frac{\partial^2 g_i}{\partial\alpha\partial\beta}-(1-2\nu)\int_z^\infty \frac{\partial^2 g_i}{\partial\alpha\partial\beta}\,\mathrm{d}z\right\}-\frac{1}{2}\Bigg\{c_2\frac{\partial g_1}{\partial\beta}+c_3\frac{\partial g_1}{\partial\alpha}\\
&\qquad +\int_z^\infty\!\!\int_z^\infty\left(\frac{\partial^2}{\partial\beta^2}-\frac{\partial^2}{\partial\alpha^2}\right)\left(c_3\frac{\partial g_1}{\partial\alpha}-c_2\frac{\partial g_1}{\partial\beta}\right)\mathrm{d}^2z\Bigg\}\Bigg]^*,\\
\tau_{yz} &= \frac{-1}{2\pi}\left[-\sum_i c_i z\frac{\partial^2 g_i}{\partial\beta\partial z}+c_3\frac{\partial g_1}{\partial z}\right]^*,\\
\tau_{zx} &= \frac{-1}{2\pi}\left[-\sum_i c_i z\frac{\partial^2 g_i}{\partial\alpha\partial z}+c_2\frac{\partial g_1}{\partial z}\right]^*,\\
\frac{Eu}{2(1+\nu)} &= \frac{-1}{4\pi}\left[\sum_i c_i\left\{-z\frac{\partial g_i}{\partial\alpha}+(1-2\nu)\int_z^\infty\frac{\partial g_i}{\partial\alpha}\,\mathrm{d}z\right\}+c_2g_1+\int_z^\infty\!\!\int_z^\infty\frac{\partial}{\partial\beta}\left(c_3\frac{\partial g_1}{\partial\alpha}-c_2\frac{\partial g_1}{\partial\beta}\right)\mathrm{d}^2z\right]^*,\\
\frac{Ev}{2(1+\nu)} &= \frac{-1}{4\pi}\left[\sum_i c_i\left\{-z\frac{\partial g_i}{\partial\beta}+(1-2\nu)\int_z^\infty\frac{\partial g_i}{\partial\beta}\,\mathrm{d}z\right\}+c_3g_1+\int_z^\infty\!\!\int_z^\infty\frac{\partial}{\partial\alpha}\left(c_2\frac{\partial g_1}{\partial\beta}-c_3\frac{\partial g_1}{\partial\alpha}\right)\mathrm{d}^2z\right]^*,\\
\frac{Ew}{2(1+\nu)} &= \frac{-1}{4\pi}\left[\sum_i c_i\left\{z\frac{\partial g_i}{\partial z}-2(1-\nu)g_i\right\}-\int_z^\infty\left(c_2\frac{\partial g_1}{\partial\alpha}+c_3\frac{\partial g_1}{\partial\beta}\right)\mathrm{d}z\right]^*.
\end{aligned}
\right\}\quad (21)
$$

Here E is Young's modulus, ν is Poisson's ratio, the stress components are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}, and the displacement components are u, v, w.

The algebra required to complete the derivation of formulas for the stress and displacement components is long enough so that it is reasonable to record some intermediate steps. In particular it involves integrations with respect to z over an infinite range, several of which require summation of corner evaluations as in eqn (12) to secure convergence. A table of evaluations for key infinite integrals is provided in Appendix B. Another key relation is that if $z \geq 0$ (as it is in the half space), then†

$$\theta_3 = \theta_1 + \theta_2. \tag{22}$$

These relationships aid in evaluating the many derivatives and integrals of g_1, g_2 and g_3 appearing in the eqn (21). Forms of those derivatives and integrals suitable for use in the eqns

†Trigonometric identities and simple algebra show

$$\arctan\frac{\beta z}{\alpha\rho}+\arctan\frac{\alpha z}{\beta\rho}=\arctan\frac{z\rho}{\alpha\beta}=\frac{\pi}{2}\operatorname{sgn}\alpha\beta z-\arctan\frac{\alpha\beta}{z\rho}.$$

Thus

$$
\begin{aligned}
\theta_1+\theta_2 &= \left(\arctan\frac{\beta}{\alpha}+\arctan\frac{\alpha}{\beta}\right)-\left(\arctan\frac{\beta z}{\alpha\rho}+\arctan\frac{\alpha z}{\beta\rho}\right)\\
&= \frac{\pi}{2}\operatorname{sgn}\alpha\beta-\frac{\pi}{2}\operatorname{sgn}\alpha\beta z+\arctan\frac{\alpha\beta}{z\rho}=\frac{\pi}{2}\operatorname{sgn}\alpha\beta(1-\operatorname{sgn}z)+\theta_3.
\end{aligned}
$$

Here sgn $X = -1$ or 1 according as $X < 0$ or $X \geq 0$.

(21) can thus be found and they are tabulated in Appendix C. Combining these results leads to formulas for the stresses and displacements in these forms:

$$\begin{gathered}\sigma_x=\sum_{i=1}^{3} c_i K_i^{xx}(x,y,z), \quad \sigma_y=\sum_{i=1}^{3} c_i K_i^{yy}(x,y,z), \quad \sigma_z=\sum_{i=1}^{3} c_i K_i^{zz}(x,y,z),\\ \tau_{xy}=\sum_{i=1}^{3} c_i K_i^{xy}(x,y,z), \quad \tau_{yz}=\sum_{i=1}^{\infty} c_i K_i^{yz}(x,y,z), \quad \tau_{zx}=\sum_{i=1}^{3} c_i K_i^{zx}(x,y,z),\\ 2\mu u=\sum_{i=1}^{3} c_i K_i^{u}(x,y,z), \quad 2\mu v=\sum_{i=1}^{3} c_i K_i^{v}(x,y,z), \quad 2\mu w=\sum_{i=1}^{3} c_i K_i^{w}(x,y,z).\end{gathered} \tag{23}$$

Here μ is the shear modulus, that is $\mu = E/[2(1+\nu)]$, also

$$K_i^{xx}(x,y,z)=-\frac{1}{2\pi}[H_i^{xx}(\alpha,\beta,z)]^*,\ K_i^{yy}(x,y,z)=-\frac{1}{2\pi}[H_i^{yy}(\alpha,\beta,z)]^*, \tag{24}$$

and so forth, for K_i^{zz}, K_i^{xy}, ---, K_i^{w}, where the functions H_i^{xx} and so forth are:

$$\left.\begin{aligned}
H_1^{xx}&=-2z\rho+\alpha z\ln(\alpha+\rho)+2\beta z\ln(\beta+\rho)+\alpha\beta\theta_3+(1-2\nu)\left\{\frac{z\rho}{2}-\alpha z\ln(\alpha+\rho)\right.\\
&\quad\left.-\frac{\alpha^2-\beta^2}{2}\ln(z+\rho)-\alpha\beta\theta_2\right\}\\
H_2^{xx}&=\frac{1}{2}\alpha\rho-\frac{3(\beta^2-z^2)}{2}\ln(\alpha+\rho)-2\alpha\beta\ln(\beta+\rho)+3\beta z\theta_3+(1-2\nu)\left\{\frac{\alpha\rho}{2}\right.\\
&\quad\left.+\frac{\beta^2-z^2}{2}\ln(\alpha+\rho)-\alpha z\ln(z+\rho)-\beta z\theta_2\right\}\\
H_3^{xx}&=\beta\rho-\alpha\beta\ln(\alpha+\rho)+z^2\ln(\beta+\rho)+\alpha z\theta_3+(1-2\nu)\left\{-\beta\rho+\alpha\beta\ln(\alpha+\rho)\right.\\
&\quad\left.+\beta z\ln(z+\rho)-\alpha z\theta_2\right\}
\end{aligned}\right\} \tag{25a}$$

$$\left.\begin{aligned}
H_1^{yy}&=-2z\rho+2\alpha z\ln(\alpha+\rho)+\beta z\ln(\beta+\rho)+\alpha\beta\theta_3+(1-2\nu)\left\{\frac{z\rho}{2}-\beta z\ln(\beta+\rho)\right.\\
&\quad\left.+\frac{\alpha^2-\beta^2}{2}\ln(z+\rho)-\alpha\beta\theta_1\right\}\\
H_2^{yy}&=\alpha\rho+z^2\ln(\alpha+\rho)-\alpha\beta\ln(\beta+\rho)+\beta z\theta_3+(1-2\nu)\left\{-\alpha\rho+\alpha\beta\ln(\beta+\rho)\right.\\
&\quad\left.+\alpha z\ln(z+\rho)-\beta z\theta_1\right\}\\
H_3^{yy}&=\frac{1}{2}\beta\rho-2\alpha\beta\ln(\alpha+\rho)-\frac{3(\alpha^2-z^2)}{2}\ln(\beta+\rho)+3\alpha z\theta_3\\
&\quad+(1-2\nu)\left\{\frac{\beta\rho}{2}+\frac{\alpha^2-z^2}{2}\ln(\beta+\rho)-\beta z\ln(z+\rho)-\alpha z\theta_1\right\}
\end{aligned}\right\} \tag{25b}$$

$$\left.\begin{aligned}
H_1^{zz}&=z\rho+\alpha\beta\theta_3\\
H_2^{zz}&=-z^2\ln(\alpha+\rho)-\beta z\theta_3\\
H_3^{zz}&=-z^2\ln(\beta+\rho)-\alpha z\theta_3
\end{aligned}\right\} \tag{25c}$$

$$\left.\begin{aligned}
H_1^{xy} &= \beta z\ln(\alpha+\rho)+\alpha z\ln(\beta+\rho)-z^2\theta_3+(1-2\nu)\Big\{\beta z\ln(\alpha+\rho)+\alpha z\ln(\beta+\rho)\\
&\quad+\alpha\beta\ln(z+\rho)+\frac{1}{2}\alpha^2\theta_1+\frac{1}{2}\beta^2\theta_2-\frac{z^2}{2}\theta_3\Big\}\\
H_2^{xy} &= \beta\rho-\alpha\beta\ln(\alpha+\rho)+z^2\ln(\beta+\rho)+\alpha z\theta_3+(1-2\nu)\Big\{-\frac{\beta\rho}{2}-\frac{\alpha^2-z^2}{2}\ln(\beta+\rho)\\
&\quad+\beta z\ln(z+\rho)+\alpha z\theta_1\Big\}\\
H_3^{xy} &= \alpha\rho+z^2\ln(\alpha+\rho)-\alpha\beta\ln(\beta+\rho)+\beta z\theta_3+(1-2\nu)\Big\{-\frac{\alpha\rho}{2}-\frac{\beta^2-z^2}{2}\ln(\alpha+\rho)\\
&\quad+\alpha z\ln(z+\rho)+\beta z\theta_2\Big\}
\end{aligned}\right\}\quad(25d)$$

$$\left.\begin{aligned}
H_1^{yz} &= z^2\ln(\beta+\rho)+\alpha z\theta_3\\
H_2^{yz} &= -\beta z\ln(\alpha+\rho)-\alpha z\ln(\beta+\rho)+z^2\theta_3\\
H_3^{yz} &= 2z\rho-2\alpha z\ln(\alpha+\rho)-\beta z\ln(\beta+\rho)-\alpha\beta\theta_3
\end{aligned}\right\}\quad(25e)$$

$$\left.\begin{aligned}
H_1^{zx} &= z^2\ln(\alpha+\rho)+\beta z\theta_3\\
H_2^{zx} &= 2z\rho-\alpha z\ln(\alpha+\rho)-2\beta z\ln(\beta+\rho)-\alpha\beta\theta_3\\
H_3^{zx} &= -\beta z\ln(\alpha+\rho)-\alpha z\ln(\beta+\rho)+z^2\theta_3
\end{aligned}\right\}\quad(25f)$$

$$\left.\begin{aligned}
H_1^{u} &= \frac{\alpha z\rho}{2}-\frac{z(\beta^2-z^2)}{2}\ln(\alpha+\rho)-\alpha\beta z\ln(\beta+\rho)+\beta z^2\theta_3+(1-2\nu)\Big\{\frac{\alpha z\rho}{3}\\
&\quad-\frac{z(3\beta^2-z^2)}{6}\ln(\alpha+\rho)-\alpha\beta z\ln(\beta+\rho)+\frac{\alpha(\alpha^2-3\beta^2)}{6}\ln(z+\rho)-\frac{\alpha^2\beta\theta_1}{2}\\
&\quad-\frac{\beta^3\theta_2}{6}+\frac{\beta z^2\theta_3}{2}\Big\}\\
H_2^{u} &= -\frac{(\beta^2-2z^2)\rho}{2}+\alpha(\beta^2-z^2)\ln(\alpha+\rho)+\frac{\beta(\alpha^2-3z^2)}{2}\ln(\beta+\rho)-2\alpha\beta z\theta_3\\
&\quad+(1-2\nu)\Big\{-\frac{\alpha^2\rho}{2}+\frac{\rho^3}{6}+\frac{\beta(\alpha^2-z^2)}{2}\ln(\beta+\rho)+\frac{z(\alpha^2-\beta^2)}{2}\ln(z+\rho)-\alpha\beta z\theta_1\Big\}\\
H_3^{u} &= -\frac{\alpha\beta\rho}{3}-\frac{\beta(\beta^2+3z^2)}{6}\ln(\alpha+\rho)-\frac{\alpha(\alpha^2+3z^2)}{6}\ln(\beta+\rho)+\frac{z^3}{3}\theta_3+(1-2\nu)\Big\{+\frac{\alpha\beta\rho}{3}\\
&\quad+\frac{\beta(\beta^2-3z^2)}{6}\ln(\alpha+\rho)+\frac{\alpha(\alpha^2-3z^2)}{6}\ln(\beta+\rho)-\alpha\beta z\ln(z+\rho)-\frac{\alpha^2z\theta_1}{2}\\
&\quad-\frac{\beta^2z\theta_2}{2}+\frac{z^3\theta_3}{6}\Big\}
\end{aligned}\right\}\quad(26a)$$

$$
\left.
\begin{aligned}
H_1^{v} &= \frac{\beta z\rho}{2} - \alpha\beta z\ln(\alpha+\rho) - \frac{z(\alpha^2-z^2)}{2}\ln(\beta+\rho) + \alpha z^2\theta_3 + (1-2\nu)\left\{\frac{\beta z\rho}{3}\right. \\
&\quad - \alpha\beta z\ln(\alpha+\rho) - \frac{z(3\alpha^2-z^2)}{6}\ln(\beta+\rho) - \frac{\beta(3\alpha^2-\beta^2)}{6}\ln(z+\rho) \\
&\quad \left. - \frac{\alpha^3\theta_1}{6} - \frac{\alpha\beta^2\theta_2}{2} + \frac{\alpha z^2\theta_3}{2}\right\} \\
H_2^{v} &= -\frac{\alpha\beta\rho}{3} - \frac{\beta(\beta^2+3z^2)}{6}\ln(\alpha+\rho) - \frac{\alpha(\alpha^2+3z^2)}{6}\ln(\beta+\rho) + \frac{z^3}{3}\theta_3 + (1-2\nu)\left\{\frac{\alpha\beta\rho}{3} + \right. \\
&\quad + \frac{\beta(\beta^2-3z^2)}{6}\ln(\alpha+\rho) + \frac{\alpha(\alpha^2-3z^2)}{6}\ln(\beta+\rho) - \alpha\beta z\ln(z+\rho) \\
&\quad \left. - \frac{\alpha^2 z\theta_1}{2} - \frac{\beta^2 z\theta_2}{2} + \frac{z^3\theta_3}{6}\right\} \\
H_3^{v} &= -\frac{(\alpha^2-2z^2)\rho}{2} + \frac{\alpha(\beta^2-3z^2)}{2}\ln(\alpha+\rho) + \beta(\alpha^2-z^2)\ln(\beta+\rho) - 2\alpha\beta z\theta_3 \\
&\quad + (1-2\nu)\left\{-\frac{\beta^2\rho}{2} + \frac{\rho^3}{6} + \frac{\alpha(\beta^2-z^2)}{2}\ln(\alpha+\rho) - \frac{z(\alpha^2-\beta^2)}{2}\ln(z+\rho) - \alpha\beta z\theta_2\right\}
\end{aligned}
\right\} \quad \text{(26b)}
$$

$$
\left.
\begin{aligned}
H_1^{w} &= \frac{\rho^3}{6} + \frac{z^2\rho}{2} - \frac{\alpha(\beta^2+z^2)}{2}\ln(\alpha+\rho) - \frac{\beta(\alpha^2+z^2)}{2}\ln(\beta+\rho) + (1-2\nu)\left\{\frac{\rho^3}{6} - \frac{z^2\rho}{2}\right. \\
&\quad \left. - \frac{\alpha(\beta^2-z^2)}{2}\ln(\alpha+\rho) - \frac{\beta(\alpha^2-z^2)}{2}\ln(\beta+\rho) + \alpha\beta z\theta_3\right\} \\
H_2^{w} &= -\frac{\alpha z\rho}{2} + \frac{z(\beta^2-z^2)}{2}\ln(\alpha+\rho) + \alpha\beta z\ln(\beta+\rho) - \beta z^2\theta_3 + (1-2\nu)\left\{\frac{\alpha z\rho}{3}\right. \\
&\quad - \frac{z(3\beta^2-z^2)}{6}\ln(\alpha+\rho) - \alpha\beta z\ln(\beta+\rho) + \frac{\alpha(\alpha^2-3\beta^2)}{6}\ln(z+\rho) - \frac{\alpha^2\beta\theta_1}{2} \\
&\quad \left. - \frac{\beta^3\theta_2}{6} + \frac{\beta z^2\theta_3}{2}\right\} \\
H_3^{w} &= -\frac{\beta z\rho}{2} + \alpha\beta z\ln(\alpha+\rho) + \frac{z(\alpha^2-z^2)}{2}\ln(\beta+\rho) - \alpha z^2\theta_3 + (1-2\nu)\left\{\frac{\beta z\rho}{3} - \alpha\beta z\ln(\alpha+\rho)\right. \\
&\quad \left. - \frac{z(3\alpha^2-z^2)}{6}\ln(\beta+\rho) - \frac{\beta(3\alpha^2-\beta^2)}{6}\ln(z+\rho) - \frac{\alpha^3\theta_1}{6} - \frac{\alpha\beta^2\theta_2}{2} + \frac{\alpha z^2\theta_3}{2}\right\} .
\end{aligned}
\right\} \quad \text{(26c)}
$$

Termwise examination of these functions H_1^{xx} through H_3^{w} shows the disposition of discontinuities which might have remained in the influences of the three load constants on the stresses and displacements. The function $\ln(\alpha+\rho)$, which occurs frequently, becomes singular if $\beta = z = 0$ and $\alpha \leq 0$, but each occurrence of it is multiplied by a factor involving β and/or z in such a way that the product vanishes in the limit as β and z vanish. The singularities in $\ln(\beta+\rho)$ which arise where $\alpha = z = 0$ and $\beta \leq 0$ are likewise obliterated by factors involving α and/or z. The singularities in $\ln(z+\rho)$ which arise where α, β and z vanish are obliterated by factors involving α and/or β. Jumps may occur in the arctangents for θ_1 as α passes through zero, but in all but one occurrence of θ_1 these jumps are obliterated through multiplication by α. The lone exception is the term $\beta z\theta_1$ in H_2^{yy}, but here θ_1 itself is continuous as α passes through zero if $z > 0$, and if $z = 0$ the product $\beta z\theta_1$ vanishes and is still continuous though α passes through zero. Similar jumps which may occur in the arctangents for θ_2 as β passes through zero are obliterated by a factor β except in the term $\alpha z\theta_2$ in H_3^{xx}, but there the vanishing of θ_2 or z assures continuity as β passes through zero. The arctangent for θ_3 varies continuously if $z > 0$, but if $z \to 0$ then $\theta_3 \to (\pi/2)\,\mathrm{sgn}\,(\alpha\beta)$ which jumps as $\alpha\beta$ passes through zero. However, all occurrences of θ_3 in H_1^{xx} through H_3^{w} include either a factor z or a factor $\alpha\beta$, so again each term remains continuous. Since the logarithms and arctangents which have been considered here are the only possible sources of discontinuities in the stresses and displacements due to pyramidal surface loads on the half space, and those discontinuities have been shown to be obliterated, it follows that those stresses and displacements vary continuously in the half space.

OBSERVATIONS CONCERNING USE OF PYRAMIDAL LOADS

Remarks on computing procedures

The influence functions K_i^{xx}, $K_i^{yy}, \ldots, K_i^{w}$, as presented in eqns (24)–(26), are sums of elementary mathematical functions, so they are readily calculable, but they are voluminous enough to profit from good organization, especially when effects from many pyramidal loads are to be combined. Some simplifying features are apparent, such as the repetitive use of the few transcendental functions $\ln(\alpha+\rho)$, $\ln(\beta+\rho)$, $\ln(z+\rho)$, θ_1, θ_2, θ_3, the repetition of blocks of terms such as some in H_3^{xx} and H_2^{xy} and the symmetries with respect to α and β as illustrated in H_3^{xx} and H_3^{yy}. Somewhat less obvious is that when contributions to stresses at a given point (x, y, z) are to be combined for contributions from overlapping pyramids some of the same corner evaluations (as required by eqn 12) are used for several pyramidal loads. To employ these latter repetitions advantageously, it is helpful to compute corner evaluations associated with all the pyramidal bases before adding those for any one pyramid, then to get the weighted sum as in eqn (12) by selecting the corners for each pyramid.

A computing program (called SURFAC) was prepared to assemble the sums required to compute stresses and displacements from the combined action of many pyramids. It does this first in terms of unassigned peak loads so that boundary condition equations can be assembled to be used in finding proper peak loads when that is needed, but when the peak loads are assigned the program can be used for complete evaluation of the stresses and displacements. In doing this it uses indices assigned to all the lattice points associated with pyramidal bases and employs a table showing the indices for the nine points for each pyramidal base. The assembly of this table has been automated, using a program called LATTICE, so that the entire calculation can proceed rapidly and conveniently. These programs have been used for the illustrative calculations included here for effects from pyramidal loads.

With loads assigned over only one pyramidal base (that is at only one pyramid point), typical calculations yielding all the stress and displacement components at any one point in the body were found to use about 0.036 seconds of central processor time in a CDC Cyber 73 computer. When 25 or more interrelated bases were used, this unit rate was reduced by a factor 3 or more.

Comparison of stresses from pyramidal and rectangular elemental loads

A major reason for introducing the pyramidal load elements was to avoid stress discontinuities inherent in the use of load elements utilizing uniform loads with jumps between the elements. In order to compare stresses from pyramidal loads with stresses for example from elements having uniform loads over rectangles, it is necessary to find what stresses arise from those uniform loads. To this end some formulas for stresses from uniform rectangular loads, as abstracted from Ref. [7], are shown in Appendix D. Other formulations for these stresses could be drawn from Love[3] or Smith and Alavi[4], but those from Ref. [7] are more directly comparable with the pyramidal formulations since that reference indeed offered an early approach to the pyramidal formulations.

Comparison of formulas for stresses from uniform rectangular loads, as shown in Appendix D, with corresponding formulas related to pyramidal loads, as given in eqns (23)–(25), show that the potentially singular logarithms which were obliterated by extra factors in the eqns (25) are not thus obliterated with the uniform rectangular loads. Furthermore the jumps in θ_1, θ_2 and θ_3 which are obliterated in the eqns (25) are not obliterated in stresses from uniform rectangular loads. Thus stresses associated with steps in patterns developed with uniform rectangular load elements entail many distortions of stresses near the surface which are avoided if pyramidal elements are used.

In order to illustrate how great the differences may be between stresses developed by these alternative systems, consider the component σ_x associated with unit shear loads $s(\sim\tau_{zx})$ of the alternative kinds acting over the area where $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. Figure 4 shows the variation of this stress in two planes (the surface of the half space and the vertical plane where $y = 0$) as due to the pyramidal load, while Fig. 5 shows corresponding patterns due to the uniform rectangular load. It can be seen that σ_x from the pyramidal load varies gently and continuously, but with the uniform rectangular load it varies extensively and indeed has a whole line of singularities. If stresses near the surface of the half space are a significant issue, then the differences between the two kinds of load elements are important and the pyramidal element

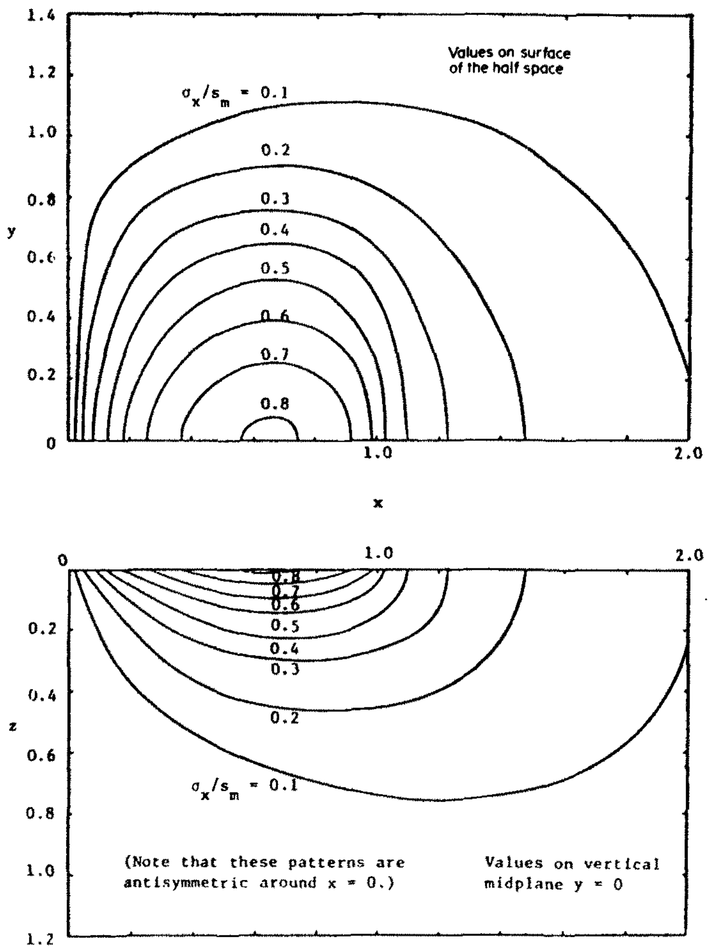


Fig. 4. Stresses σ_x due to unit pyramidal shear $s(\sim \tau_{zx})$ acting over $|x| \leq 1$, $|y| \leq 1$.

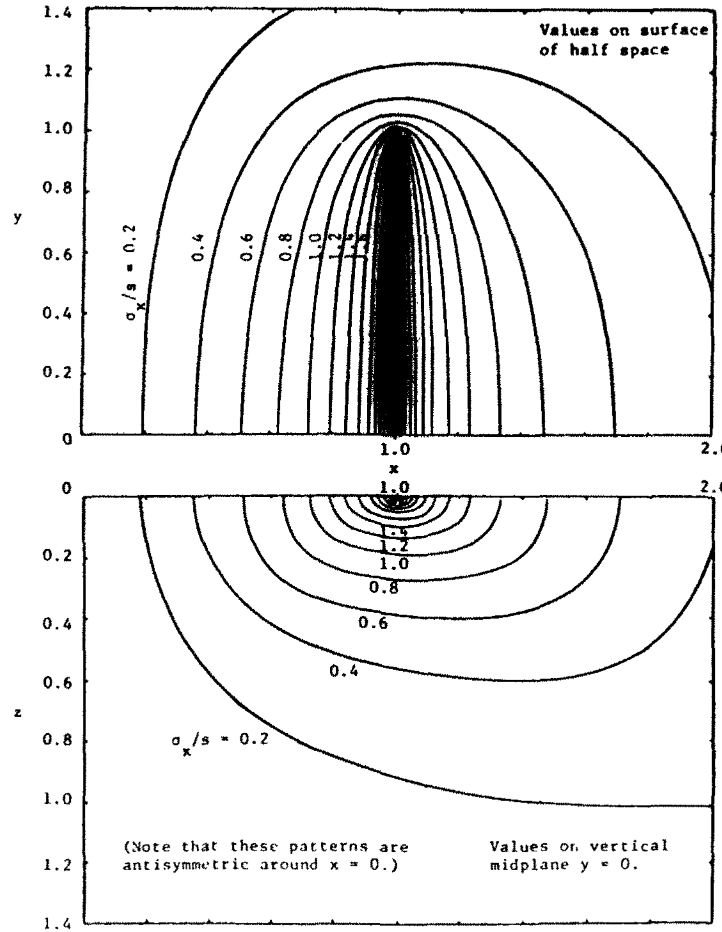
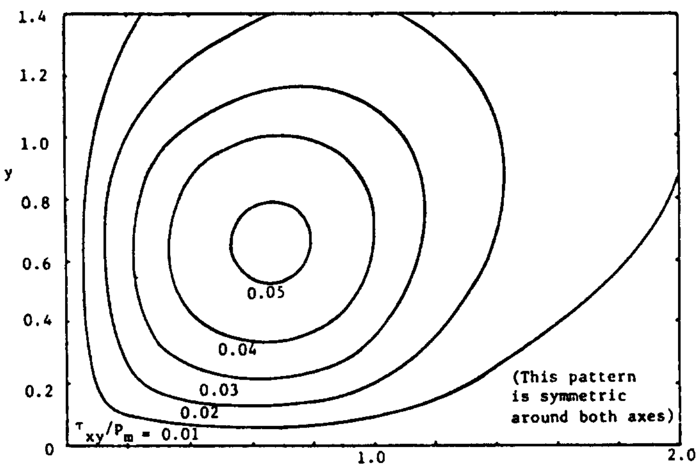


Fig. 5. Stresses σ_x due to unit uniform rectangular shear $s(\sim \tau_{zx})$ acting over $|x| \leq 1$, $|y| \leq 1$.

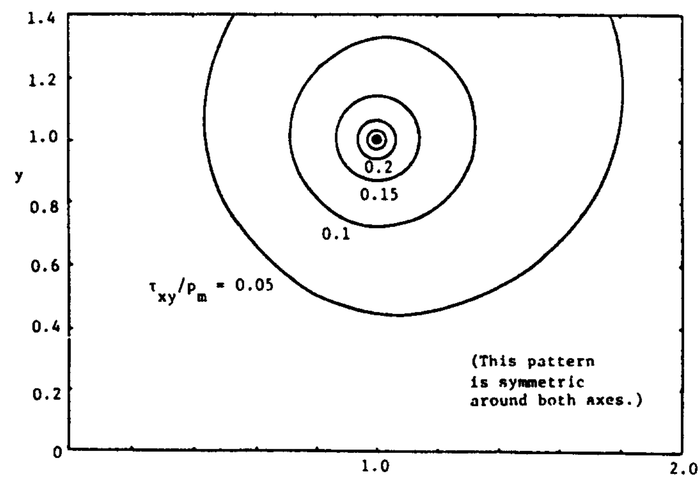
must be conceded to yield more appropriate stress calculations. A further illustration of a singularity associated with a uniform rectangular element, though not with the corresponding pyramidal element, is shown in Fig. 6, which shows the component τ_{xy} on surface of the half space due to unit loads $p(\sim -\sigma_z)$ of the alternative kinds acting over the area where $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. This singularity from the uniform rectangular normal load is more restricted than that associated with the uniform shear load, but it yet yields an infinite value at the corners of the rectangle. The corresponding stresses τ_{xy} generated by the pyramidal form of load p are finite and much gentler.

Illustrative calculations from multiple pyramidal loads

To illustrate use of multiple pyramidal loads in a practical situation, consider stresses generated by contact of a cylinder on a half space. The usual assumption that the contact area is infinitely long is awkward for pyramidal loads, but it can be approximated by making the contact area long. The contact load varies elliptically in the short direction, so taking the half width of contact as the unit distance, the usual form of the load is $p = p_0\sqrt{(1-x^2)}$. Figure 7 shows an approximation of this load pattern by pyramidal loads, putting pyramidal peaks where $y = \pm 20$ and $x = 0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8, \pm 0.9, \pm 0.95$, with respective values of p/p_0 being 1.00, 0.98, 0.92, 0.81, 0.61, 0.46, 0.348, so that the load per unit length in the main section is nearly $\pi/2$. The far ends of the pattern are placed arbitrarily at $y = \pm 21$. In order to introduce shearing loads, one may assume frictional force proportional to p in the x-direction.



a. Stresses τ_{xy} on z-plane due to unit pyramidal load p.



b. Stresses τ_{xy} on z-plane due to unit uniform rectangular load p.

Fig. 6. Stresses τ_{xy} due to alternative normal loads $p(\sim -\sigma_z)$ acting over $|x| \leq 1$, $|y| \leq 1$.

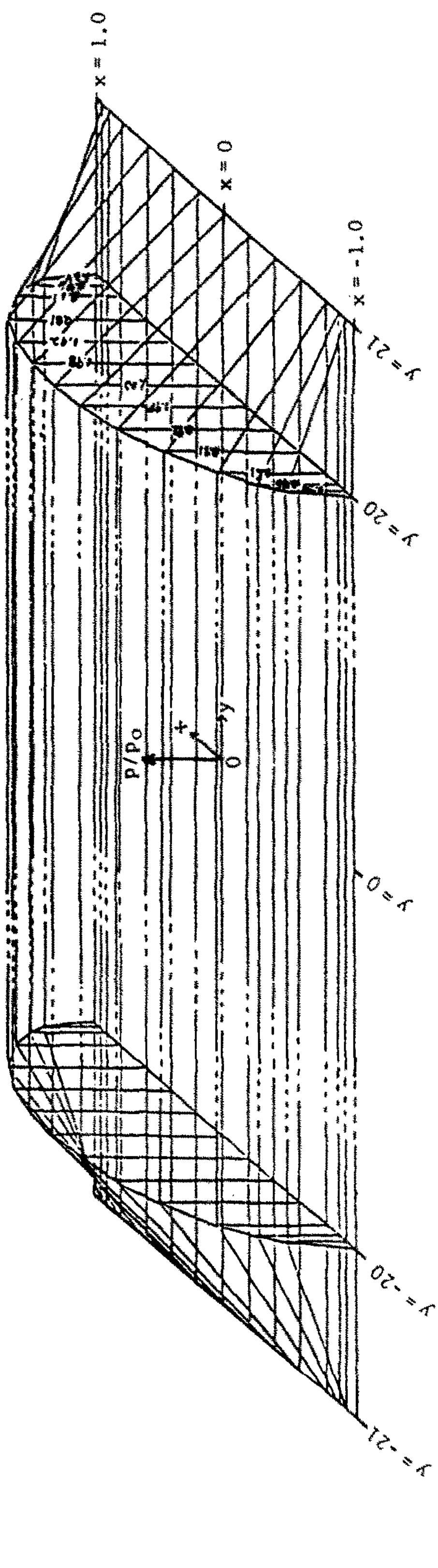


Fig. 7. Approximation of normal load under long cylindrical contact using pyramidal load elements.

The program SURFAC was used to calculate stress components due to the load pattern of Fig. 7 at nearly 400 points on the midplane $y = 0$, including the effect of friction with coefficient 0.5. From these it further found values for the second stress invariant J_2, that is

$$J_2 = \left\{\frac{1}{6}[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + (\sigma_x - \sigma_y)^2] + \tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2\right\}^{1/2}.$$

The variations of J_2/p_0 thus found at several depths z below the surface of the half space are shown in Fig. 8. Since the stresses from an infinitely long semielliptical load have been found analytically by Poritsky, it is possible to compare the values in Fig. 8 with exact values for the nearly equivalent ideal case, so the ideal values for $z = 0$ are shown as a dashed line in Fig. 8. As can be seen, the values from SURFAC are nearly like the ideal values, but there is a little waviness added by the coarseness of the approximation of the surface load. The waviness is damped rapidly as the depth z increases. The small departures at large values of x reflect the use of a finitely long contact area, so that agreement with the ideal case at remote points was not fully achieved even at $y = 0$. These discrepancies from the ideal case are trivial, however, in comparison with departures that rectangular load elements would have given, for with them the singularities in σ_x and σ_y at each jump of the frictional load would have produced 14 infinite spikes in the curve for $z = 0$.

From the results shown in Fig. 8, one may infer curves on the plane $y = 0$ along which J_2 has constant values, and this is done in Fig. 9. The curves correspond to some computed by Hamilton and Goodman[9] using Poritsky's exact solution. The places where their curves differ noticeably from those derived from the pyramidal load theory are shown by dashed lines in Fig. 9. The discrepancies are small, and some may be mere irregularities of drawing, so again the agreement with the ideal solution is good, despite some sharp local details in the stress patterns. Use of more and longer pyramidal loads of course could improve the results from the pyramidal load theory slightly.

As a final example of use of multiple pyramidal loads, consider a freeing load pattern found while performing a stress analysis for a surface crack in a plate[1]. The crack was part-circular, with depth-to-length ratio 0.25, it penetrated 0.7 of the plate thickness, and the total stress on it was to equal a uniform normal load p_0. The effects of loads associated with the crack were expressed in terms of a large new family of crack functions[10, 11]. In the balancing of effects of crack loads and surface-freeing loads, the distribution of normal loads needed on the cracked surface of the plate was found to be that shown in Fig. 10. This figure illustrates the power of the pyramidal loads to represent a complicated pattern smoothly. The effects of the crack loads as felt on the plate surface are potentially very complex, but an important issue in the crack analysis is how well the overall surface boundary conditions are satisfied. The use of the continuously varying surface loads makes extensive checking of the overall satisfaction of those conditions feasible. Also, since shearing components of the freeing loads are also needed, it avoids the introduction of a vast pattern of singularities among stress components on the surface where much care is needed for the analysis. It is possible to perform surface crack analyses with stepwise surface load elements, but the use of the continuously varying pyramidal loads greatly simplifies the problems of satisfying boundary conditions, including logical problems.

It may be added that crack-analysis freeing load patterns such as that in Fig. 10 have no direct mechanical significance, they are merely complements to effects from crack functions which may be chosen in many ways. In this light, the fine adaptability of pyramidal load representations derived from their automatic interpolation helps greatly in fitting boundary conditions at points between those where fitting is done directly.

CONCLUSIONS

A new form of elemental load has been introduced for use in stress analyses for a half space, and formulas have been derived for the implied stress and displacement patterns presuming loads applied either normally or tangentially. The load elements have a pyramidal form which imparts continuity and piecewise linear variation in two directions to resultant loads from many elements. Sums of these elements are well suited to representation of arbitrary load

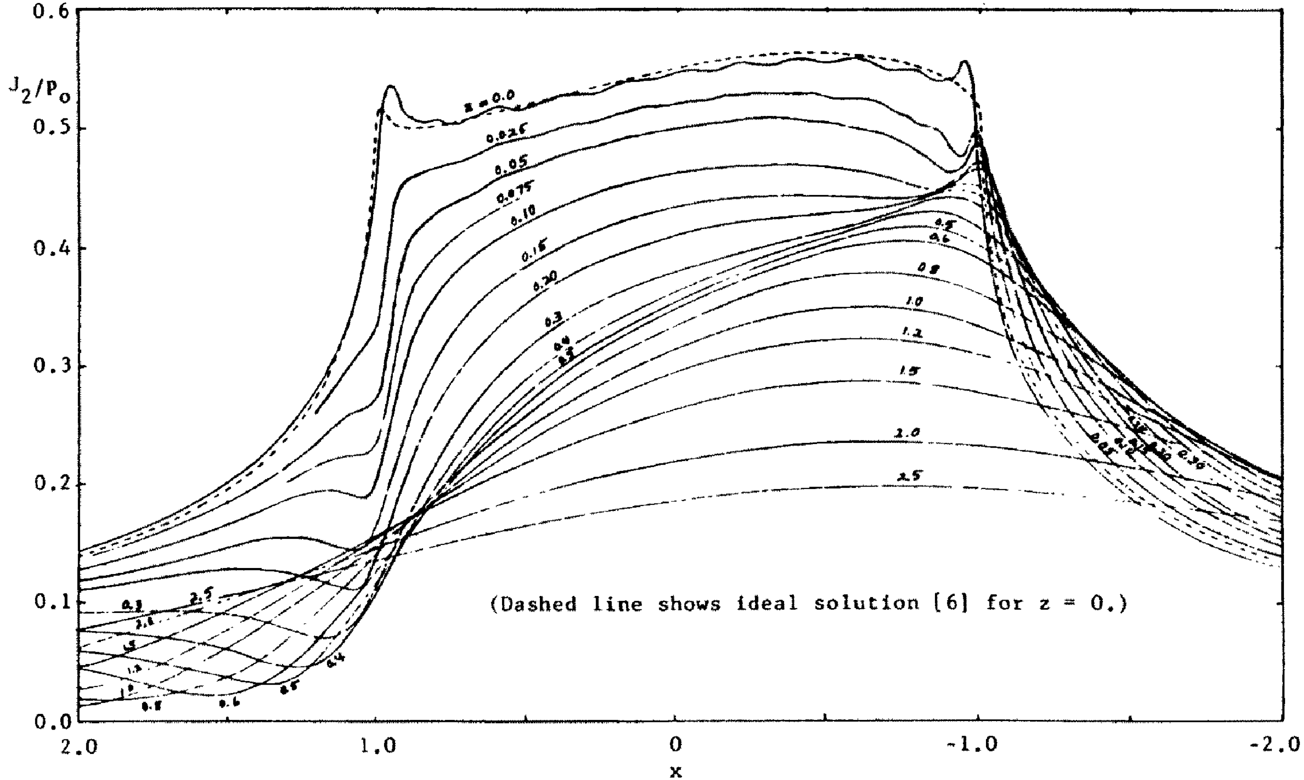


Fig. 8. Values of stress invariant J_2 at various levels below cylindrical contact with superposed frictional force ($\sim \tau_{zx}$) with coefficient 0.5.

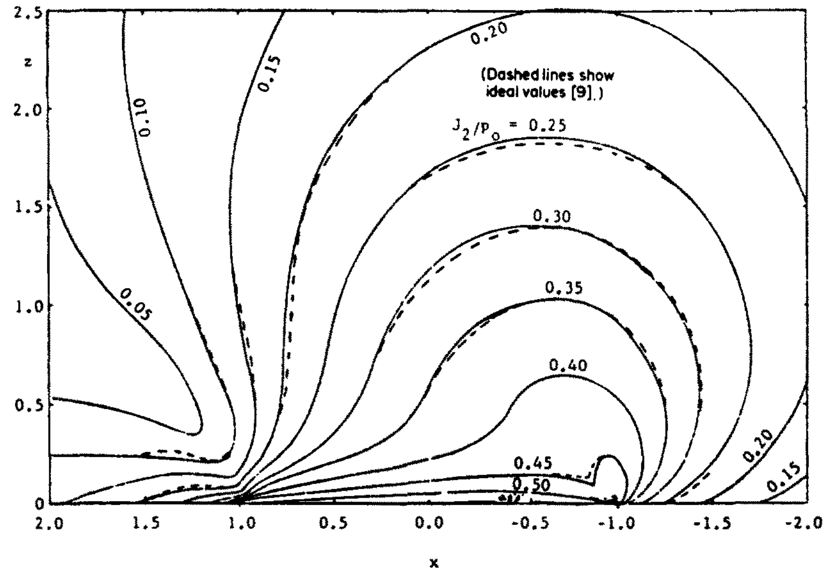


Fig. 9. Comparative contours for J_2/p_0 from cylindrical contact, including frictional force ($\sim \tau_{zx}$) with coefficient 0.5.

patterns. The formulas for the stresses and displacements, as given by eqns (23)–(26) are somewhat voluminous because of the many components and load elements treated, but they are expressed solely in terms of elementary functions and all traces of discontinuous stresses are obliterated by using limiting values of individual terms. Use of the formulas implies considerable arithmetic, but the organization of that arithmetic through use of the summation introduced by eqn (12) leads to efficient calculation processes for a computer.

A comparison of stresses induced by elemental pyramidal loads or by uniform rectangular loads, as illustrated by Figs. 4 and 5, shows the greatly improved regularity associated with the pyramidal loads. A comparison with ideal solutions for stresses due to cylindrical contact with superposed tangential force from friction illustrates the efficiency of the pyramidal elements in

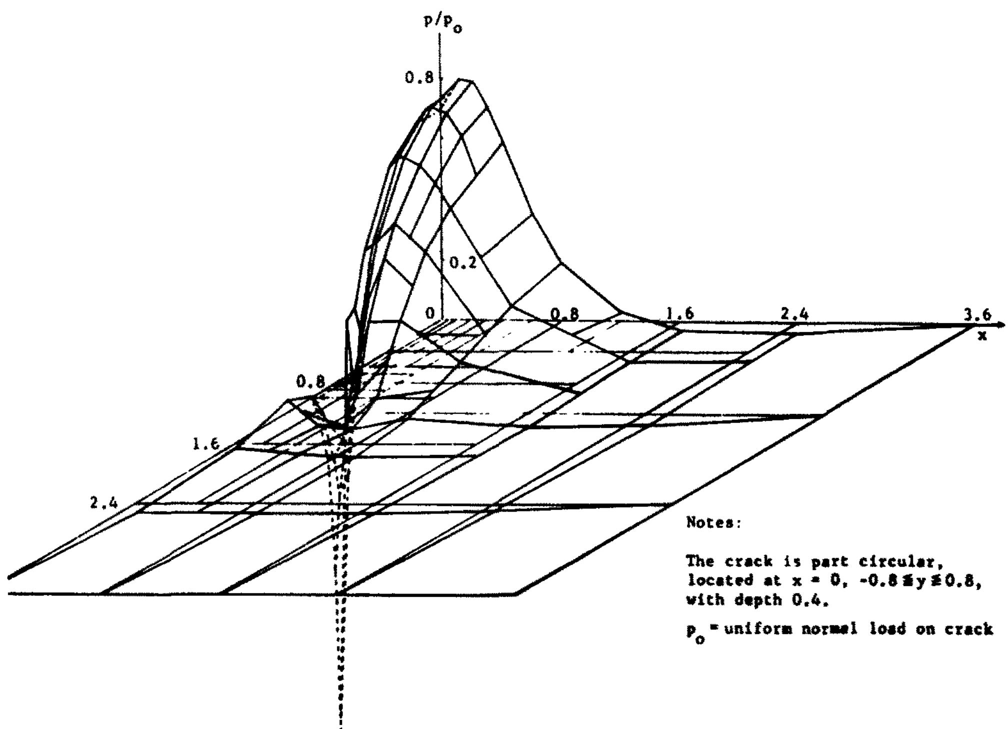


Fig. 10. Representative normal freeing load for a surface crack analysis.

determining stress distributions, as in Figs. 8 and 9. Finally, brief consideration of freeing stresses for a crack analysis, as illustrated by Fig. 10, shows that the adaptability and continuity of multiple pyramidal loads offer logical as well as computational help in performing the troublesome analysis for surface cracks.

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APPENDIX A

Some one-variable indefinite integrals

Integration formulas used in deriving the integrals $g_{ij}(\alpha, \beta, z)$ include the following ones which are not found in standard tabulations

$$\int \frac{\sqrt{(x^2+a^2+b^2)}}{x^2+a^2}\,dx = \ln\frac{\sqrt{(x^2+a^2+b^2)}+x}{\sqrt{(a^2+b^2)}} + \frac{b}{a}\arctan\frac{bx}{a\sqrt{(x^2+a^2+b^2)}} + c, \quad \text{if} \quad a \neq 0.$$

$$\int \frac{\sqrt{(x^2+a^2+b^2)}}{(x^2+a^2)^2}\,dx = \frac{x\sqrt{(x^2+a^2+b^2)}}{2a^2(x^2+a^2)} + \frac{a^2+b^2}{2a^3b}\arctan\frac{bx}{a\sqrt{(x^2+a^2+b^2)}} + c, \quad \text{if} \quad a \neq 0.$$

$$\int \frac{x\sqrt{(x^2+a^2+b^2)}}{x^2+a^2}\,dx = \sqrt{(x^2+a^2+b^2)} + \frac{b}{2}\ln\left[\frac{\sqrt{(x^2+a^2+b^2)}-b}{\sqrt{(x^2+a^2+b^2)}+b}\right] + c.$$

$$\int \frac{x^2\sqrt{(x^2+a^2+b^2)}}{(x^2+a^2)^2}\,dx = \ln\left[\frac{\sqrt{(x^2+a^2+b^2)}+x}{\sqrt{(a^2+b^2)}}\right] - \frac{x\sqrt{(x^2+a^2+b^2)}}{2(a^2+x^2)} - \frac{a^2-b^2}{2ab}\arctan\frac{bx}{a\sqrt{(x^2+a^2+b^2)}} + c, \quad \text{if} \quad a \neq 0.$$

These formulas were given in Ref. [5].

APPENDIX B

Some useful infinite integrals

The following integrals are useful in deriving formulas for stress and displacement components. Note that all these formulas remain valid if α and β are interchanged together with θ_1 and θ_2, and if m and n are assigned any real values. Moreover, the fourth, fifth, sixth, eighth, and tenth would hold without the summation implied by the operator $[\ldots]^*$, that is with α and β being treated as being arbitrary. The quantities ρ, θ_1, θ_2 and θ_3 are those of the eqns (8).

$$\int_z^\infty [\rho]^*\,dz = \left[-\frac{z\rho}{2} - \frac{\alpha^2+\beta^2}{2}\ln(z+\rho)\right]^*$$

$$\int_z^\infty [z\rho]^*\,dz = \left[-\tfrac{1}{3}\rho^3\right]^*$$

$$\int_z^\infty \left[\frac{\alpha^m}{\rho}\right]^* dz = [-\alpha^m \ln(\alpha+\rho)]^*$$

$$\int_z^\infty \left[\frac{\alpha^m\beta^n}{\rho^3}\right]^* dz = \left[\frac{\alpha^m\beta^n}{\alpha^2+\beta^2}\left(1-\frac{z}{\rho}\right)\right]^*$$

$$\int_z^\infty \left[\frac{\alpha^m\beta^n}{\alpha^2+\beta^2}\left(1-\frac{z}{\rho}\right)\right]^* dz = \left[\frac{\alpha^m\beta^n}{z+\rho}\right]^*$$

$$\int_z^\infty \left[\frac{\alpha^m\beta^n}{(\alpha^2+z^2)\rho}\right]^* dz = [\alpha^{m-1}\beta^{n-1}\theta_1]^*$$

$$\int_z^\infty \left[\frac{\alpha^m\beta z}{(\alpha^2+z^2)\rho}\right]^* dz = [\alpha^m \ln(\beta+\rho)]^*$$

$$\int_z^\infty \left[\frac{2\alpha^m\beta^n}{(\alpha^2+z^2)^2\rho}\right]^* dz = \left[\alpha^{m-2}\beta^{n-2}\left(1-\frac{z}{\rho}\right)-\frac{\alpha^{m-2}\beta^n z}{(\alpha^2+z^2)\rho}+\alpha^{m-3}\beta^{n-3}(\beta^2-\alpha^2)\theta_1\right]^*$$

$$\int_z^\infty [\alpha^m\theta_1]^* dz = [\alpha^{m+1}\ln(\beta+\rho)-\alpha^m z\theta_1]^*$$

$$\int_z^\infty \left[\frac{\alpha\beta}{\rho}-z\theta_3\right]^* dz = \left[\frac{1}{2}\alpha^2\theta_1+\frac{1}{2}\beta^2\theta_2-\frac{\alpha\beta}{2}+\frac{z^2}{2}\theta_3\right]^*$$

$$\int_z^\infty \left[\frac{\alpha\beta}{2}-\frac{z^2}{2}\theta_3\right]^* dz = \left[\frac{\alpha\beta\rho}{3}+\frac{z^3}{6}\ln(\beta+\rho)+\frac{\beta^3}{6}\ln(\alpha+\rho)\right]^*$$

$$\int_z^\infty [\ln(\alpha+\rho)]^* dz = [-\alpha\ln(z+\rho)-z\ln(\alpha+\rho)-\beta\theta_2]^*$$

$$\int_z^\infty [z\ln(z+\rho)]^* dz = \left[\frac{-\alpha\rho}{2}-\frac{\beta^2+z^2}{2}\ln(\alpha+\rho)\right]^*$$

$$\int_z^\infty [\alpha^m\ln(z+\rho)]^* dz = [\alpha^m\rho-\alpha^m z\ln(z+\rho)]^*$$

$$\int_z^\infty \left[\frac{\alpha\beta}{\rho}-\beta\ln(\alpha+\rho)\right]^* dz = [\beta^2\theta_2+\beta z\ln(\alpha+\rho)]^*$$

$$\int_z^\infty \left[\frac{\alpha\rho}{2}-\frac{\beta^2+z^2}{2}\ln(\alpha+\rho)\right]^* dz = \left[-\frac{\alpha z\rho}{6}-\frac{\alpha^3}{3}\ln(z+\rho)+\frac{z(3\beta^2+z^2)}{6}\ln(\alpha+\rho)+\frac{1}{3}\beta^3\theta_2\right]^*$$

These formulas can be derived from those in Appendix A of Ref.[7].

APPENDIX C

Terms arising in stresses and displacements

The following derivatives and integrals appear in the eqns (21) for the stresses and displacements due to pyramidal surface loads. For the derivatives, parts which would vanish during evaluation implied by the operator $[\ldots]^*$ [as defined by eqn (12)] have been omitted, since they drop from the eqns (21). Further functions needed for the eqns (21) can be found from those included here by simultaneously interchanging α with β, θ_1 with θ_2, and g_2 with g_3.

$$g_1(\alpha,\beta,z) = \frac{\alpha}{2}(\beta^2-z^2)\ln(\alpha+\rho)+\frac{\beta}{2}(\alpha^2-z^2)\ln(\beta+\rho)-\alpha\beta z\theta_3+\frac{z^2}{2}\rho-\frac{1}{6}\rho^3$$

$$\frac{\partial g_1}{\partial\alpha} = \frac{1}{2}(\beta^2-z^2)\ln(\alpha+\rho)+\alpha\beta\ln(\beta+\rho)-\beta z\theta_3-\frac{\alpha\rho}{2}$$

$$\frac{\partial g_1}{\partial z} = -\alpha z\ln(\alpha+\rho)-\beta z\ln(\beta+\rho)-\alpha\beta\theta_3+z\rho$$

$$\frac{\partial^2 g_1}{\partial\alpha^2} = \beta\ln(\beta+\rho)-\rho$$

$$\frac{\partial^2 g_1}{\partial z^2} = -\alpha\ln(\alpha+\rho)-\beta\ln(\beta+\rho)+2\rho$$

$$\frac{\partial^2 g_1}{\partial\alpha\partial\beta} = \beta\ln(\alpha+\rho)+\alpha\ln(\beta+\rho)-z\theta_3$$

$$\frac{\partial^2 g_1}{\partial\alpha\partial z} = -z\ln(\alpha+\rho)-\beta\theta_3$$

$$\int_z^\infty \left[\frac{\partial g_1}{\partial\alpha}\right]^* dz = \left[\frac{z(-3\beta^2+z^2)}{6}\ln(\alpha+\rho)-\alpha\beta z\ln(\beta+\rho)+\frac{\alpha(\alpha^2-3\beta^2)}{6}\ln(z+\rho)-\frac{\alpha^2\beta}{2}\theta_1-\frac{\beta^3}{6}\theta_2+\frac{\beta z^2}{2}\theta_3+\frac{\alpha z\rho}{3}\right]^*$$

$$\int_z^\infty \left[\frac{\partial^2 g_1}{\partial\alpha^2}\right]^* dz = \left[-\beta z\ln(\beta+\rho)+\frac{\alpha^2-\beta^2}{2}\ln(z+\rho)-\alpha\beta\theta_1+\frac{z\rho}{2}\right]^*$$

$$\int_z^\infty \left[\frac{\partial^2 g_1}{\partial\alpha\partial\beta}\right]^* dz = \left[-\beta z\ln(\alpha+\rho)-\alpha z\ln(\beta+\rho)-\alpha\beta\ln(z+\rho)-\frac{\alpha^2}{2}\theta_1-\frac{\beta^2}{2}\theta_2+\frac{z^2}{2}\theta_3\right]^*$$

$$\int_z^\infty \int_z^\infty \left[\frac{\partial^2 g_1}{\partial\alpha^2}\right]^* \mathrm{d}^2 z = \left[-\frac{\beta(\alpha^2-z^2)}{2}\ln(\beta+\rho) - \frac{z(\alpha^2-\beta^2)}{2}\ln(z+\rho) + \alpha\beta z\theta_1 + \frac{\alpha^2}{2}\rho - \frac{\rho^3}{6}\right]^*$$

$$\int_z^\infty \int_z^\infty \left[\frac{\partial^2 g_1}{\partial\alpha\partial\beta}\right]^* \mathrm{d}^2 z = \left[-\frac{\beta(\beta^2-3z^2)}{6}\ln(\alpha+\rho) - \frac{\alpha(\alpha^2-3z^2)}{6}\ln(\beta+\rho) + \alpha\beta z\ln(z+\rho) + \frac{\alpha^2 z}{2}\theta_1 + \frac{\beta^2 z}{2}\theta_2 - \frac{z^3}{6}\theta_3 - \frac{\alpha\beta\rho}{3}\right]^*$$

$$\int_z^\infty \int_z^\infty \left[\frac{\partial^3 g_1}{\partial\alpha^3}\right]^* \mathrm{d}^2 z = [-\alpha\beta\ln(\beta+\rho) - \alpha z\ln(z+\rho) + \beta z\theta_1 + \alpha\rho]^*$$

$$\int_z^\infty \int_z^\infty \left[\frac{\partial^3 g_1}{\partial\alpha^2\partial\beta}\right]^* \mathrm{d}^2 z = \left[-\frac{\alpha^2-z^2}{2}\ln(\beta+\rho) + \beta z\ln(z+\rho) + \alpha z\theta_1 - \frac{\beta\rho}{2}\right]^*$$

$$g_2(\alpha,\beta,z) = \frac{z}{6}(3\beta^2-z^2)\ln(\alpha+\rho) + \alpha\beta z\ln(\beta+\rho) - \frac{\alpha}{6}(\alpha^2-3\beta^2)\ln(z+\rho) + \frac{\alpha^2\beta}{2}\theta_1 + \frac{\beta^3}{6}\theta_2 - \frac{\beta z^2}{2}\theta_3 - \frac{\alpha z\rho}{3} - \frac{\alpha\beta^2}{6}$$

$$\frac{\partial g_2}{\partial\alpha} = \beta z\ln(\beta+\rho) - \frac{\alpha^2-\beta^2}{2}\ln(z+\rho) + \alpha\beta\theta_1 - \frac{z\rho}{2}$$

$$\frac{\partial g_2}{\partial\beta} = \beta z\ln(\alpha+\rho) + \alpha z\ln(\beta+\rho) + \alpha\beta\ln(z+\rho) + \frac{\alpha^2}{2}\theta_1 + \frac{\beta^2}{2}\theta_2 - \frac{z^2}{2}\theta_3$$

$$\frac{\partial g_2}{\partial z} = \frac{\beta^2-z^2}{2}\ln(\alpha+\rho) + \alpha\beta\ln(\beta+\rho) - \beta z\theta_3 - \frac{\alpha\rho}{2}$$

$$\frac{\partial^2 g_2}{\partial\alpha^2} = -\alpha\ln(z+\rho) + \beta\theta_1$$

$$\frac{\partial^2 g_2}{\partial\beta^2} = z\ln(\alpha+\rho) + \alpha\ln(z+\rho) + \beta\theta_2$$

$$\frac{\partial^2 g_2}{\partial z^2} = -z\ln(\alpha+\rho) - \beta\theta_3$$

$$\frac{\partial^2 g_2}{\partial\alpha\partial\beta} = z\ln(\beta+\rho) + \beta\ln(z+\rho) + \alpha\theta_1$$

$$\frac{\partial^2 g_2}{\partial\alpha\partial z} = \beta\ln(\beta+\rho) - \rho$$

$$\frac{\partial^2 g_2}{\partial\beta\partial z} = \beta\ln(\alpha+\rho) + \alpha\ln(\beta+\rho) - z\theta_3$$

$$\int_z^\infty \left[\frac{\partial g_2}{\partial\alpha}\right]^* \mathrm{d}z = \left[\frac{\beta(\alpha^2-z^2)}{2}\ln(\beta+\rho) + \frac{z(\alpha^2-\beta^2)}{2}\ln(z+\rho) - \alpha\beta z\theta_1 - \frac{\alpha^2}{2}\rho + \frac{1}{6}\rho^3\right]^*$$

$$\int_z^\infty \left[\frac{\partial g_2}{\partial\beta}\right]^* \mathrm{d}z = \left[\frac{\beta(\beta^2-3z^2)}{6}\ln(\alpha+\rho) + \frac{\alpha(\alpha^2-3z^2)}{6}\ln(\beta+\rho) - \alpha\beta z\ln(z+\rho) - \frac{\alpha^2 z}{2}\theta_1 - \frac{\beta^2 z}{2}\theta_2 + \frac{z^3}{6}\theta_3 + \frac{\alpha\beta\rho}{3}\right]^*$$

$$\int_z^\infty \left[\frac{\partial^2 g_2}{\partial\alpha^2}\right]^* \mathrm{d}z = [-\alpha\rho + \alpha z\ln(z+\rho) + \alpha\beta\ln(\beta+\rho) - \beta z\theta_1]^*$$

$$\int_z^\infty \left[\frac{\partial^2 g_2}{\partial\alpha\partial\beta}\right]^* \mathrm{d}z = \left[\frac{\alpha^2-z^2}{2}\ln(\beta+\rho) - \beta z\ln(z+\rho) - \alpha z\theta_1 + \frac{\beta\rho}{2}\right]^*$$

$$\int_z^\infty \left[\frac{\partial^2 g_2}{\partial\beta^2}\right]^* \mathrm{d}z = \left[\frac{\beta^2-z^2}{2}\ln(\alpha+\rho) - \alpha z\ln(z+\rho) - \beta z\theta_2 + \frac{\alpha\rho}{2}\right]^*.$$

APPENDIX D

Formulas for stresses and displacements due to uniform rectangular surface loads

Let load elements be chosen by which a pressure load c_1 and shear loads c_2 and c_3 are applied, corresponding to those of eqn (18), but now uniformly over a rectangular base. Then, as may be inferred from Ref. [7], the resulting stresses and displacements may be expressed again by equations of the form (23), but with the influence functions K_i^{xx} and so forth replaced by new ones of the forms

$$K_i^{xx}(x,y,z) = -\frac{1}{2\pi}[G_{i1}^{xx}(\alpha,\beta,z)]_c,\ K_i^{yy}(x,y,z) = -\frac{1}{2\pi}[G_{i1}^{yy}(\alpha,\beta,z)]_c,$$

and so forth. The evaluations implied by the subscript c are made using the four corner pairs of values for α and β at the corners of the rectangle. Thus, letting the corners be where $x = x_l$ or x_u and $y = y_l$ and y_u, so that $\alpha_l = x_l - x$, $\alpha_u = x_u - x$, $\beta_l = y_l - y$, $\beta_u = y_u - y$, one takes

$$[G(\alpha,\beta,z)]_c = G(\alpha_u,\beta_u,z) - G(\alpha_l,\beta_u,z) + G(\alpha_l,\beta_l,z) - G(\alpha_u,\beta_l,z).$$

Using the same notations ρ, θ_1, θ_2 and θ_3 as with pyramidal loads, the appropriate functions to use with uniform rectangular loads are, from Ref. [7]:

$$G_{11}^{xx}(\alpha,\beta,z) = -\frac{\alpha\beta z}{(\alpha^2+z^2)\rho} + \theta_3 - (1-2\nu)\theta_2$$

$$G^{xx}_{21}(\alpha, \beta, z) = -\frac{\alpha^2\beta}{(\alpha^2+z^2)} - 2\ln(\beta+\rho) + (1-2\nu)\frac{\beta}{z+\rho}$$

$$G^{xx}_{31}(\alpha, \beta, z) = \frac{\alpha}{\rho} - \ln(\alpha+\rho) - (1-2\nu)\left\{\frac{\alpha}{z+\rho} - \ln(\alpha+\rho)\right\}$$

$$G^{yy}_{11}(\alpha, \beta, z) = -\frac{\alpha\beta z}{(\beta^2+z^2)\rho} + \theta_3 - (1-2\nu)\theta_1$$

$$G^{yy}_{21}(\alpha, \beta, z) = \frac{\beta}{\rho} - \ln(\beta+\rho) - (1-2\nu)\left\{\frac{\beta}{z+\rho} - \ln(\beta+\rho)\right\}$$

$$G^{yy}_{31}(\alpha, \beta, z) = -\frac{\alpha\beta^2}{(\beta^2+z^2)\rho} - 2\ln(\alpha+\rho) + (1-2\nu)\frac{\alpha}{z+\rho}$$

$$G^{zz}_{11}(\alpha, \beta, z) = \frac{\alpha\beta z}{(\alpha^2+z^2)\rho} + \frac{\alpha\beta z}{(\beta^2+z^2)\rho} + \theta_3$$

$$G^{zz}_{21}(\alpha, \beta, z) = -\frac{\beta z^2}{(\alpha^2+z^2)\rho}$$

$$G^{zz}_{31}(\alpha, \beta, z) = -\frac{\alpha z^2}{(\beta^2+z^2)\rho}$$

$$G^{yz}_{11}(\alpha, \beta, z) = \frac{\alpha z^2}{(\beta^2+z^2)\rho}$$

$$G^{yz}_{21}(\alpha, \beta, z) = -\frac{z}{\rho}$$

$$G^{yz}_{31}(\alpha, \beta, z) = \frac{\alpha\beta z}{(\beta^2+z^2)\rho} - \theta_3$$

$$G^{zx}_{11}(\alpha, \beta, z) = \frac{\beta z^2}{(\alpha^2+z^2)\rho}$$

$$G^{zx}_{21}(\alpha, \beta, z) = \frac{\alpha\beta z}{(\alpha^2+z^2)} - \theta_3$$

$$G^{zx}_{31}(\alpha, \beta, z) = -\frac{z}{\rho}$$

$$G^{xy}_{11}(\alpha, \beta, z) = \frac{z}{\rho} + (1-2\nu)\ln(z+\rho)$$

$$G^{xy}_{21}(\alpha, \beta, z) = \frac{\alpha}{\rho} - \ln(\alpha+\rho) - (1-2\nu)\frac{\alpha}{z+\rho}$$

$$G^{xy}_{31}(\alpha, \beta, z) = \frac{\beta}{\rho} - \ln(\beta+\rho) - (1-2\nu)\frac{\beta}{z+\rho}$$

$$G^{u}_{11}(\alpha, \beta, z) = -z\ln(\beta+\rho) - (1-2\nu)\{\beta\ln(z+\rho) + z\ln(\beta+\rho) + \alpha\theta_1\}$$

$$G^{u}_{21}(\alpha, \beta, z) = 2\beta\ln(\alpha+\rho) + \alpha\ln(\beta+\rho) - 2z\theta_3 - (1-2\nu)\{-\alpha\ln(\beta+\rho) + z\theta_1\}$$

$$G^{u}_{31}(\alpha, \beta, z) = -\rho - (1-2\nu)\{-\rho + z\ln(z+\rho)\}$$

$$G^{v}_{11}(\alpha, \beta, z) = -z\ln(\alpha+\rho) - (1-2\nu)\{\alpha\ln(z+\rho) + z\ln(\alpha+\rho) + \beta\theta_2\}$$

$$G^{v}_{21}(\alpha, \beta, z) = -\rho - (1-2\nu)\{-\rho + z\ln(z+\rho)\}$$

$$G^{v}_{31}(\alpha, \beta, z) = 2\alpha\ln(\beta+\rho) + \beta\ln(\alpha+\rho) - 2z\theta_3 - (1-2\nu)\{-\beta\ln(\alpha+\rho) + z\theta_2\}$$

$$G^{w}_{11}(\alpha, \beta, z) = -\beta\ln(\alpha+\rho) - \alpha\ln(\beta+\rho) - (1-2\nu)\{\beta\ln(\alpha+\rho) + \alpha\ln(\beta+\rho) - z\theta 3\}$$

$$G^{w}_{21}(\alpha, \beta, z) = z\ln(\beta+\rho) - (1-2\nu)\{\beta\ln(z+\rho) + z\ln(\beta+\rho) + \alpha\theta_1\}$$

$$G^{w}_{31}(\alpha, \beta, z) = z\ln(\alpha+\rho) - (1-2\nu)\{\alpha\ln(z+\rho) + z\ln(\alpha+\rho) + \beta\theta_2\}.$$

These influence functions for uniform rectangular loads involve several singularities and jumps. Thus the function $\ln(\alpha+\rho)$ which occurs in G^{xx}_{31}, G^{yy}_{31} and G^{xy}_{21} is singular where $\beta = z = 0$ and $\alpha \leq 0$. The function $\ln(\beta+\rho)$ which occurs in G^{xx}_{21}, G^{yy}_{21} and G^{xy}_{31} is singular where $\alpha = z = 0$ and $\beta \leq 0$, and $\ln(z+\rho)$ in G^{xy}_{11} is singular where α, β and z vanish. The functions θ_1, θ_2 and θ_3 have many jumps on the plane where $z = 0$. In their effects on stresses, these singularities and jumps persist here, not being obliterated by extra factors as they are in influence functions for pyramidal loads. There are also many indeterminacies which appear in the influences of rectangular loads affecting stresses. The discontinuities appearing in expressions for displacements due to the uniform rectangular loads are obliterated by factors multiplying them, as is reasonable, but as noted the stresses from such loads involve many complications which do not arise with pyramidal loads.

Reference[7], from which these expressions for effects of rectangular loads are drawn, also treats the broader case of stresses and displacments due to loads varying linearly in both directions over a rectangle, as illustrated by the central part of Fig. 1(a). This approach provides an alternative approach to getting continuous approximations to arbitrarily distributed surface loads on a half space, but it is far more cumbersome than the use of pyramidal load elements.